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Elementary Abelian 3-Subgroups of the Monster

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We classify the maximal elementary abelian 3-subgroups of the Monster simple group. There are 17 conjugacy classes of such subgroups. Fifteen of the classes have groups of order 3^7, and the other classes have groups of order 3^6 and 3^8. The classification uses the construction of 3-local subgroups of the Monster from [3].

1. INTRODUCTION

In this paper we classify the maximal elementary abelian 3-subgroups of the Monster simple group, which we denote $\mathbb{M}$. Computations used in the classification use the explicit construction of 3-local subgroups of $\mathbb{M}$ from [3]. Previously, in [4] we classified all elementary abelian 5-subgroups of $\mathbb{M}$.

Using the notation of [1] for conjugacy classes, let $E_1 < E_2 < E_3$ be elementary abelian 3-subgroups whose nonidentity elements are in class $3B$, with E_i of order 3^i and $N_{\mathbb{M}}(E_i)$ a maximal subgroup of $\mathbb{M}$. Let $C = N_{\mathbb{M}}(E_1)$, $N = N_{\mathbb{M}}(E_2)$, and $L = N_{\mathbb{M}}(E_3)$. Then C, N, and L are groups of shapes $3_+^{1+12} \cdot 2Suz{:}2$, $3^{2+5+10} \cdot (M_{11} \times GL(2,3))$, and $3^{3+2+6+6} \cdot (S_2 \times SL(2,3))$, respectively, where Suz is the sporadic Suzuki group and S_2 is isomorphic to a Sylow 2-subgroup of the Mathieu group M_{11}. We will define $Q = O_3(C)$, $P = O_3(N)$, and $R = O_3(L)$, where $O_3(H)$ denotes the largest normal 3-subgroup of a group H.

The classification is divided into four parts. In Section 3, we show that every maximal elementary abelian 3-subgroup is conjugate to a subgroup of $P{:}T$, where T is a Sylow 3-subgroup of M_{11}. In Sections 4 and 5, we classify the maximal elementary abelian subgroups that contain a conjugate

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of $Z(R)$. In Section 6, we determine the N-classes of maximal elementary abelian subgroups that do not contain an N-conjugate of $Z(R)$, for which purpose we develop the idea of the δ-shape of a subgroup. In Section 7, we determine the fusion in $\mathbb{M}$ of the groups that do not contain an N-conjugate of $Z(R)$, both among themselves and with groups containing an N-conjugate of $Z(R)$.

Our main theorem is:

THEOREM 1.1. *There are* 17 *conjugacy classes of maximal elementary abelian* 3*-subgroups of the Monster group. Elements of just* 13 *of the classes contain a conjugate of* $Z(R)$. *Representatives of the classes and their types are listed in Tables* VI *and* VIII.

It follows from the classification that one class has elements with rank 8, one class has elements with rank 6, and the remaining classes have elements with rank 7. Furthermore, the class of rank 6 is the only one whose groups contain elements of class 3C.

2. PRELIMINARY MATERIAL

We summarize the essentials of the construction of specific 3-local subgroups of $\mathbb{M}$ from [3]. These groups are constructed using the ternary Golay code $\mathscr{G}$.

2.1. *Ternary Golay Code*

The ternary Golay code ($\mathscr{G}$) is a code of dimension 6, length 12, and minimum weight 6 over the field of order 3. We use a modified version of the minimog construction of the ternary Golay code in [1].

DEFINITION 2.1. The ternary Golay code $\mathscr{G}$ is the span of the following vectors of $\mathbb{F}_3^{12}$:

$$\mathbf{f} = (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0) \qquad \mathbf{g} = (1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0)$$

$$\mathbf{x} = (0, 0, 0, 0, 1, 2, 0, 1, 2, 0, 2, 1) \qquad \mathbf{y} = (0, 1, 2, 0, 0, 0, 0, 2, 1, 0, 2, 1)$$

$$\mathbf{z} = (1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0) \qquad \mathbf{u} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).$$

Each code word in the minimog construction corresponds to the code word in $\mathscr{G}$ formed by multiplying its first and last columns by -1. The minimog array

$$c = \boxed{\begin{array}{cccc} c_1 & c_4 & c_7 & c_{10} \\ c_2 & c_5 & c_8 & c_{11} \\ c_3 & c_6 & c_9 & c_{12} \end{array}}$$

shows how we arrange the components of a typical code word. The code $\mathcal{G}$ is self-orthogonal with respect to the bilinear form $(c, d) = \sum_{i=1}^{12} c_i d_i$. We also use the trilinear form $(c, d, e) = \sum_{i=1}^{12} c_i d_i e_i$. In addition to the specific elements of $\mathcal{G}$ listed in Definition 2.1, let $\mathbf{h} = \mathbf{f} + \mathbf{g} + \mathbf{x} - \mathbf{y} + \mathbf{z}$, let $\mathbf{k} = \mathbf{f} + \mathbf{g} - \mathbf{x} + \mathbf{y} + \mathbf{z}$, and let $\mathbf{l} = -\mathbf{f} - \mathbf{g} + \mathbf{x} + \mathbf{z} - \mathbf{u}$. We display these elements of $\mathcal{G}$ in minimog format in Table I.

The automorphism group of $\mathcal{G}$ is the double cover of the Mathieu group M_{12}. This group has two conjugacy classes of subgroups isomorphic to the Mathieu group M_{11}. The M_{11} subgroup of particular interest to us is the stabilizer of the all 1's code word $\mathbf{u} \in \mathcal{G}$; this is a 3-transitive permutation group on 12 points.

DEFINITION 2.2. We define the following permutations, which are automorphisms of $\mathcal{G}$:

$$
\begin{aligned}
T_1 &= (4,5,6)(7,8,9)(10,11,12) & T_2 &= (1,2,3)(4,5,6)(7,9,8) \\
\sigma_1 &= (1,10)(2,4)(3,7)(5,8) & \sigma_2 &= (1,10)(4,12)(6,8)(7,11) \\
\chi &= (4,7)(5,9)(6,8)(11,12) & \pi &= (1,7,2,5,11)(3,6,8,12,4) \\
\rho &= (1,4)(2,5,3,6)(7,10) & \xi &= (2,8,6)(3,9,5)(4,7,10) \\
&\quad \times (8,11,9,12) & & \\
\eta &= (1,5,9)(3,11,12)(6,8,10) & \mu &= (1,10,12,7,11,2)(2,6,8)(3,9).
\end{aligned}
$$

TABLE I
Some Elements of $\mathcal{G}$

$$
\mathbf{f} = \boxed{\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}} \qquad
\mathbf{x} = \boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{matrix}} \qquad
\mathbf{z} = \boxed{\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}}
$$

$$
\mathbf{g} = \boxed{\begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix}} \qquad
\mathbf{y} = \boxed{\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 \end{matrix}} \qquad
\mathbf{u} = \boxed{\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}}
$$

$$
\mathbf{h} = \boxed{\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}} \qquad
\mathbf{k} = \boxed{\begin{matrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{matrix}} \qquad
\mathbf{l} = \boxed{\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}}
$$

It is easy to verify that these permutations are automorphisms of $\mathscr{G}$ by showing that they map the basis $\{\mathbf{f}, \mathbf{g}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}\}$ into $\mathscr{G}$. In Table II, we give diagrams that show how the permutations act with respect to the arrangement of coordinates in the minimog array.

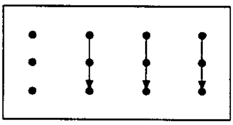
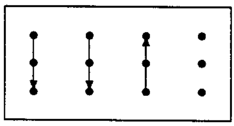
Since the code word $\mathbf{u}$ is fixed by M_{11}, the quotient $\bar{\mathscr{G}} = \mathscr{G}/\langle \mathbf{u} \rangle$ is an M_{11} module. We call the duals $\mathscr{G}^* = \mathrm{Hom}_{\mathbb{F}_3}(\mathscr{G}, \mathbb{F}_3)$ and $\bar{\mathscr{G}}^* = \mathrm{Hom}_{\mathbb{F}_3}(\bar{\mathscr{G}}, \mathbb{F}_3)$. We identify $\mathscr{G}^*$ with the quotient $\mathbb{F}_3^{12}/\mathscr{G}$, and we label elements of $\mathscr{G}^*$ by low weight representatives.

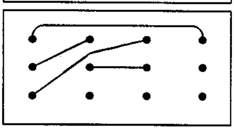
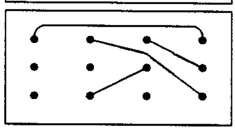
DEFINITION 2.3. Let δ_{ijk} be the element of $\mathscr{G}^*$ such that $\delta_{ijk}(c) = c_i + c_j + c_k$ for all code words c, and let δ_{i-j} be the element of $\mathscr{G}^*$ such that $\delta_{i-j}(c) = c_i - c_j$ for all code words c.

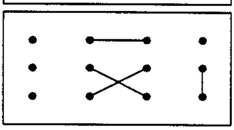
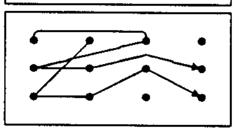
If i, j, or k has two digits, we write $\delta_{i,\,j,\,k}$ for δ_{ijk}. Note that every nonzero element of $\bar{\mathscr{G}}^*$ can be described as δ_{ijk} or δ_{i-j} for some i, j, k. Some elements of $\bar{\mathscr{G}}^*$ are determined by pairs of code words as in the next definition.

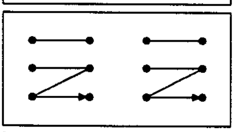
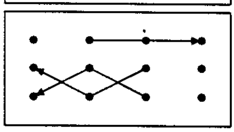
DEFINITION 2.4. Let c and d be elements of $\mathscr{G}$. We define $\delta_{c,\,d}$ to be the element of $\mathscr{G}^*$ represented by the vector $(c_1 d_1, \ldots, c_{12} d_{12})$.

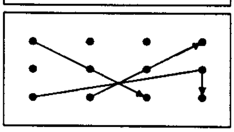
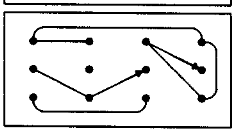
TABLE II
Some Elements of M_{11}

$T_1 =$  $T_2 =$ 

$\sigma_1 =$  $\sigma_2 =$ 

$\chi =$  $\pi =$ 

$\rho =$  $\xi =$ 

$\eta =$  $\mu =$ 

We also need the function Ψ defined below.

DEFINITION 2.5. Let $l: \mathbb{F}_3 \to \mathbb{Z}$ be defined by $l(0) = 0$, $l(1) = 1$, and $l(-1) = -1$. Define $\Psi: \mathcal{G} \to \mathbb{F}_3$ by $\Psi(c) = \frac{1}{3}\sum_{i=1}^{12} l(c_i) (\mathrm{mod}\, 3)$.

Thus $\Psi(\mathbf{f}) = -1$, $\Psi(\mathbf{x}) = 0$, and so on.

2.2. *The Loop $L(\mathcal{G})$ and Local Subgroups*

We construct a loop $L(\mathcal{G})$ from $\mathcal{G}$, and we use the loop to describe some 3-local subgroups of $\mathbb{M}$.

DEFINITION 2.6. Let $L(\mathcal{G}) = \mathbb{F}_3 \times \mathcal{G}$, and define a binary operation on $L(\mathcal{G})$ by $(a, c)(b, d) = (a + b + (c, d, d), c + d)$. Also let $¢ = (1, 0)$.

In [3], the loop $L(\mathcal{G})$ is used to construct groups N and C. The group C is isomorphic to the normalizer of a subgroup of the Monster of order 3, generated by an element of class $3B$. The group N is isomorphic to the normalizer in $\mathbb{M}$ of an elementary abelian subgroup of order 9, whose non-identity elements are all from class $3B$. The generators of N are elements ∞_c, 0_c, 1_c, and 2_c, where $c \in L(\mathcal{G})$; δ for $\delta \in \bar{\mathcal{G}}^*$; α for $\alpha \in M_{11}$; and x_t, x_n, and x_d corresponding to elements of $GL(2, 3)$,

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad d = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We may sometimes use ∞_c as an abbreviation for any of the elements $\infty_{(a, c)}$ where $a \in \mathbb{F}_3$.

DEFINITION 2.7. The map $\kappa: L(\mathcal{G}) \to L(\mathcal{G})$ is given by $(a, c)^\kappa = (-a, c)$.

LEMMA 2.1. *The generators listed above satisfy the relations*

$$i_c i_d = i_{cd} i_¢^{-(c, d, d)} \tag{1a}$$

$$[\infty_c, 0_d] = \delta_{c, d}^2 \infty_¢^{(c, c, d)} 0_¢^{(c, d, d)} \tag{1b}$$

$$i_c \delta = \delta i_c i_¢^{\delta(c)} \tag{1c}$$

$$\infty_c\, i_c = \delta_{c, c} (i + 1)_c (i + 2)_¢^{\Psi(c)} \tag{1d}$$

$$i_c^\alpha = i_{c^\alpha} \tag{1e}$$

$$\delta^{x_t} = \delta \tag{1f}$$

$$\infty_c^{x_t} = \infty_c \infty_¢^{-c_1} \tag{1g}$$

$$0_c^{x_t} = 1_c 1_¢^{-c_1} \tag{1h}$$

$$\delta^{x_n} = \delta^{-1} \tag{1i}$$

$$\infty_c^{x_n} = 0_{c^\kappa} \tag{1j}$$

$$0_c^{x_n} = \infty_{c^\kappa} \tag{1k}$$

$$\delta^{x_d} = \delta^{-1} \tag{1l}$$

$$\infty_c^{x_d} = \infty_{c^\kappa} \tag{1m}$$

$$0_c^{x_d} = 0_{c^\kappa}^{-1} \tag{1n}$$

$$0_{¢} = \infty_{\mathbf{u}}^{-1} \tag{1o}$$

The above relations are from [3]. They may be used to prove the next lemma.

DEFINITION 2.8. Let $P = O_3(N)$, let $C = C_{\mathbb{M}}(\infty_{¢})$, let $Q = O_3(C)$, let $L = N_{\mathbb{M}}(\langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle)$, and let $R = O_3(L)$.

LEMMA 2.2. *The group P is nilpotent of class 3, with central series $Z_1(P) = Z(P) = \langle \infty_{¢}, 0_{¢}\rangle$, $Z_2(P) = P' = Z(P)\langle \delta \mid \delta \in \bar{\mathcal{G}}^*\rangle$, and $P = P'\langle \infty_c, 0_c \mid c \in \mathcal{G}\rangle$. Also, N/P is isomorphic to the direct product $M_{11} \times GL(2,3)$.*

DEFINITION 2.9. Let $T = \langle T_1, T_2\rangle$, let $S_2 = \langle \rho, \chi\rangle$, and let $S_2^* = \langle \rho(x_d x_n)^2, \chi\rangle$, where T_1, T_2, ρ, and χ are the elements of M_{11} listed in Table II, and x_d and x_n are elements of the $GL(2,3)$ factor of N. Also let $S = PT\langle x_t\rangle$.

One may verify that T is a Sylow 3-subgroup of M_{11}, that S_2 is a Sylow 2-subgroup of M_{11} that normalizes T, and that S_2^* is isomorphic to S_2. We also note that $x_{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)} = (x_d x_n)^2$. The group S is a Sylow 3-subgroup of $\mathbb{M}$, and it is contained in C, N, and $N_{\mathbb{M}}(R)$. From Eq. (1) one may verify that $Z(S) = \langle \infty_{¢}\rangle$.

In [3] it is shown that $Q = \langle \infty_c, \delta, x_t \mid c \in L(\mathcal{G}), \delta \in \bar{\mathcal{G}}^*\rangle$, and Q is an extra special group of shape 3_+^{1+12}. There is a very useful relationship between Q and the complex Leech lattice. The following definition is essentially that of [2], with a different labeling of the coordinates.

DEFINITION 2.10. Let ω be a primitive complex cube root of unity, and let $\Theta = \omega - \omega^2$. The complex Leech lattice Λ is the set of all vectors $v \in \mathbb{Z}[\omega]^{12}$ such that

$$\text{There exists } m \in \mathbb{Z} \text{ such that } v_i \equiv m \pmod{\Theta} \tag{2a}$$

$$\sum_{i=1}^{12} l(c_i)v_i \equiv 0 \pmod 3 \qquad \text{for all } c \in \mathcal{G} \tag{2b}$$

$$\sum_{i=1}^{12} v_i \equiv -3m \pmod{3\Theta} \tag{2c}$$

There is a homomorphism from Q to $\Lambda/\Theta\Lambda$ such that the action of $C_N(\infty_{¢})$ on Q/Q' corresponds to the action of the monomial group of Λ.

LEMMA 2.3. *Suppose that* $\delta \in \bar{\mathcal{G}}^*$ *and* $c \in \mathcal{G}$. *The vectors* λ_δ, λ_c, *and* λ_1 *are elements of* Λ, *where* $\lambda_\delta = (3l(\delta_1), \ldots, 3l(\delta_{12}))$, $\lambda_c = (\Theta l(c_1), \ldots, \Theta l(c_{12}))$, *and* $\lambda_1 = (4, 1, \ldots, 1)$.

DEFINITION 2.11. Define $\phi: Q/Q' \to \Lambda/\Theta\Lambda$ by $\phi(\delta) = -\lambda_\delta + \Theta\Lambda$, $\phi(\infty_c) = \lambda_c + \Psi(c)\lambda_{\mathbf{u}} + \Theta\Lambda$, and $\phi(x_t) = \lambda_1 + \Theta\Lambda$.

Again, using results from [3], it may be shown that C is generated by $C_N(\infty_{¢})$, and an element B that acts on Q/Q' as the matrix $U \oplus \bar{U} \oplus \bar{U} \oplus U$ acts on $\Lambda/\Theta\Lambda$, where

$$U = \frac{1}{\Theta}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix}.$$

Furthermore, the normalizer $N_{\mathbb{M}}(\infty_{¢})$ is generated by C and $(x_n x_d)^2$, which acts on Q/Q' as complex conjugation acts on $\Lambda/\Theta\Lambda$.

LEMMA 2.4. *The group* R *has nilpotence class* 4, *and the ascending central series of* R *is* $Z_1(R) = Z(R) = \langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle$, $Z_2(R) = Z_1(R)\langle \delta_{2-3}, \delta_{11-12}\rangle$, $Z_3(R) = Z_2(R)\langle \delta_{10-7}, \delta_{10-4}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, 0_{\mathbf{f}}, 0_{\mathbf{g}}\rangle$, *and* $Z_4(R) = Z_3(R)\langle \infty_{\mathbf{x}}, \infty_{\mathbf{y}}, 0_{\mathbf{x}}, 0_{\mathbf{y}}, T_1, T_2\rangle = R$. *Furthermore*, L/R *is isomorphic to the direct product* $S_2^* \times SL(3, 3)$.

Proof. We show that $N_{\mathbb{M}}(R)/R$ is isomorphic to the direct product $S_2^* \times SL(3, 3)$. First, we note that the centralizer in $\mathbb{M}$ of $\langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle$ is equal to the centralizer in N of $\langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle$. Thus the centralizer of $\langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle$ is generated by $C_P(\delta_{123})$, T, and the subgroup $\langle \chi, \rho\rho^{\chi}\rangle$ of S_2. The group G generated by x_t, x_n, $\infty_{\mathbf{z}}$, $0_{\mathbf{z}}$, and B acts on $\langle \infty_{¢}, 0_{¢}, \delta_{123}\rangle$ as $SL(3, 3)$. Finally, we note that ρ centralizes $\langle \infty_{¢}, 0_{¢}\rangle$ and inverts δ_{123}, so L is generated by P, T, S_2, x_t, and x_n. Now R is generated by $C_P(\delta_{123})$ and T. Although the images of S_2 and G do not commute in L/R, the images of S_2^* and G do commute.

The claims about the central series of R may be proved using Eq. (1). ∎

The factor $SL(3, 3)$ is generated (modulo RS_2, at least) by the elements x_t, x_n, $\infty_{\mathbf{z}}$, $0_{\mathbf{z}}$, and the element B, which will be defined below. The elements x_t and x_n generate the $GL(2, 3)$ factor of N.

2.3. *The Action of B on Q/Z(Q)*

We may use ϕ and the matrix $U \oplus \bar{U} \oplus \bar{U} \oplus U$ to determine the action of B on Q/Q'. We describe the action of B and another nonmonomial element in Table III, where $x_{1,10}$ is used to denote $x_t\delta_{10-1}$.

TABLE III
Action of Nonmonomial Elements on Q/Q'

q	q^B	$q^{B\sigma_1 B^{-1}}$	$q^{B\pi B\sigma_1 B^{-1}}$
$x_{1,10}$	$\infty_{\mathbf{z}}^{-1}$	$x_{1,10}$	$\infty_{-\mathbf{f}-\mathbf{g}-\mathbf{u}}\delta_{3-2}$
$\infty_{\mathbf{z}}$	$x_{1,10}$	$\infty_{\mathbf{z}}$	$x_{1,10}^{-1}\infty_{\mathbf{x}-\mathbf{y}}\delta_{10-1}$
$\infty_{\mathbf{x}-\mathbf{y}}$	$\infty_{\mathbf{x}-\mathbf{y}}$	δ_{4-7}	∞_{-x}
$\infty_{\mathbf{x}}$	$\infty_{\mathbf{x}}$	$\infty_{\mathbf{x}}$	$\infty_{\mathbf{z}-\mathbf{f}-\mathbf{g}-\mathbf{u}}\delta_{2-3}$
$\infty_{\mathbf{f}+\mathbf{g}}$	$\delta_{4,7,10}^{-1}$	$\infty_{\mathbf{u}}$	$x_{1,10}^{-1}\delta_{1-10}$
$\infty_{\mathbf{f}-\mathbf{g}}$	δ_{4-7}	δ_{2-3}	$\infty_{\mathbf{g}-\mathbf{f}}$
$\delta_{4,7,10}$	$\infty_{\mathbf{f}+\mathbf{g}}$	$x_{1,10}\delta_{4,7,10}^{-1}$	$\infty_{-\mathbf{z}+\mathbf{f}+\mathbf{g}}\delta_{7-4}$
δ_{7-4}	$\infty_{\mathbf{f}-\mathbf{g}}$	$\infty_{\mathbf{y}-\mathbf{x}}$	$\infty_{-\mathbf{x}}\delta_{4-7}$
δ_{11-12}	δ_{11-12}	δ_{11-12}	$x_{1,10}^{-1}\infty_{\mathbf{x}-\mathbf{y}}\delta_{1-10}$
δ_{2-3}	δ_{2-3}	$\infty_{\mathbf{f}-\mathbf{g}}$	$\infty_{\mathbf{g}-\mathbf{f}}\delta_{12-11}$
δ_{123}	$0_{¢}^{-1}$	δ_{123}	$\infty_{\mathbf{f}+\mathbf{g}}$
$\infty_{\mathbf{u}}$	δ_{123}^{-1}	$\infty_{\mathbf{f}+\mathbf{g}}$	$\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$

2.4. *Conjugacy Classes*

Every element of $P \cap Q$ can be written uniquely as $\infty_c \delta \infty_{\mathbf{u}}^i$, where c has shape (0^{12}), $(1^6, 0^6)$, or $(1^3, 2^3, 0^6)$, $\delta \in P'$, and $i = 0$, 1, or 2. Let $q \in Q$. If $\phi(q)$ is congruent to an element λ with $\lambda \cdot \lambda = 18$, then q is in class $3A$, while if $\phi(q)$ is congruent to an element λ with $\lambda \cdot \lambda = 27$, then q is in class $3B$.

Most of the time, we want to know the class of all of the elements of the coset $q\langle \infty_{\mathbf{u}} \rangle$, and this is simpler to determine than the class of specific elements.

DEFINITION 2.12. For each $c \in \mathscr{G}$, let $\tilde{c}$ be an element of $\mathscr{G}$ such that the vector $(c_i \tilde{c}_i) \equiv 0 \pmod{\mathscr{G}}$.

If c is a word of shape $(1^6, 0^6)$, we may take $\tilde{c} = c$, while if c is a word of shape $(1^3, 2^3, 0^6)$, we may take $\tilde{c}$ to be a word of shape $(1^3, 2^3, 0^6)$ whose support is disjoint from the support of c.

LEMMA 2.5. *The conjugacy classes of elements of* $P \cap Q$ *are*: $\infty_{\mathbf{u}}$ *is in class* $3B$, $\delta i_{¢}$ *is in class* $3A$ *if* δ *corresponds to a cocode word of shape* $(1, -1, 0^{10})$ *or in class* $3B$ *if* δ *corresponds to a cocode word of shape* $(1^3, 0^9)$, *and the classes of the elements of* $\infty_c \delta \langle \infty_{\mathbf{u}} \rangle$ *are as stated in Table* IV.

We also need this partial refinement of Lemma 2.5.

TABLE IV
Classes of $\infty_c\delta\langle\infty_{\mathbf{u}}\rangle$

Shape of c	$\delta(\tilde{c})$	Number from Class $3A$	Number from Class $3B$
$(1^6, 0^6)$	0	2	1
$(1^6, 0^6)$	1 or 2	0	3
$(1^3, 2^3, 0^6)$	0	1	2
$(1^3, 2^3, 0^6)$	1 or 2	0	3

LEMMA 2.6. *The following elements are in class* $3A$:

1. $\infty_{c+u}\delta_{i-j}$, *where c has shape* $(1^6, 0^6)$ *and* $c_i = c_j$.
2. $\infty_{c-u}\delta_{i-j}$, *where c has shape* $(1^6, 0^6)$ *and* $c_i = c_j = 1$.
3. $\infty_c\delta_{i-j}$, *where c has shape* $(1^6, 0^6)$ *and* $c_i = c_j = 0$.
4. $\infty_{c+u}\delta_{i-j}$, *where c has shape* $(1^3, 2^3, 0^6)$ *and* $c_i = c_j = 2$.
5. $\infty_{c-u}\delta_{i-j}$, *where c has shape* $(1^3, 2^3, 0^6)$ *and* $c_i = c_j = 1$.
6. $\infty_c\delta_{i-j}$, *where c has shape* $(1^3, 2^3, 0^6)$, $\tilde{c}_i = \tilde{c}_j$, *and* $c_i + c_j = 0$.

The next lemma can be proved with the relations in Eq. (1).

LEMMA 2.7. *The element* $\infty_c 0_d$ *has order* 3 *if and only if* $(c, c, d) = 0 = (c, d, d)$.

Proof. By Eq. (1b) we have $0_d\infty_c = \infty_{\text{¢}}^{-(c,c,d)}0_{\text{¢}}^{-(c,d,d)}\delta_{c,d}\infty_c 0_d$, and by Eq. (1c) we have $\infty_c\delta_{c,d} = \delta_{c,d}\infty_c\infty_{\text{¢}}^{(c,c,d)}$ and $0_d\delta_{c,d} = \delta_{c,d}0_d 0_{\text{¢}}^{(c,d,d)}$. Thus

$$\begin{aligned}
(\infty_c 0_d)^3 &= \infty_c\left(\infty_{\text{¢}}^{-(c,c,d)}0_{\text{¢}}^{-(c,d,d)}\delta_{c,d}\infty_c 0_d\right)^2 0_d \\
&= \delta_{c,d}^2\infty_c^2\, 0_d\infty_c 0_d^2\infty_{\text{¢}}^{(c,c,d)}0_{\text{¢}}^{-(c,d,d)} \\
&= \delta_{c,d}^2\infty_c^2\infty_{\text{¢}}^{-(c,c,d)}0_{\text{¢}}^{-(c,d,d)}\delta_{c,d}\infty_c 0_d 0_d^2\infty_{\text{¢}}^{(c,c,d)}0_{\text{¢}}^{-(c,d,d)} \\
&= \delta_{c,d}^3\infty_c^2\infty_c 0_d 0_d^2\infty_{\text{¢}}^{-(c,c,d)}0_{\text{¢}}^{(c,d,d)} \\
&= \infty_{\text{¢}}^{-(c,c,d)}0_{\text{¢}}^{(c,d,d)}.
\end{aligned}$$

Thus $\infty_c 0_d$ has order 3 if and only if $(c, c, d) = (c, d, d) = 0$. ∎

LEMMA 2.8. *Every element of P of order* 3 *that is not in any N-conjugate of $P \cap Q$ is conjugate to either* $\infty_{\mathbf{f}-\mathbf{g}}0_{\mathbf{f}+\mathbf{g}-\mathbf{u}}$ *or* $\infty_{\mathbf{y}}0_{\mathbf{x}}$.

Proof. Using conjugacy in P, we can assume that such an element has the form $\infty_c 0_d$ for some c and $d \in \mathcal{G}$. As $0_{¢} \equiv \infty_{\mathbf{u}}^{-1}$ and $\infty_{¢} \equiv 0_{\mathbf{u}}$, we may assume that both c and d have weight 6. The previous lemma shows that $(c, c, d) = 0 = (c, d, d)$. This implies that the intersection of the supports of c and d has size 0, in which case both c and d have shape $(1^3, 2^3, 0^6)$, since $\infty_c 0_d$ is not in a conjugate of $P \cap Q$; it has size 3 and c and d are constant on the intersection; or the intersection has size 4 and looks like

$$c = (1, 1, -1, -1, \ldots)$$
$$d = (1, -1, 1, -1, \ldots).$$

In the third case, $\infty_c 0_d$ is conjugate to $\infty_{\mathbf{y}} 0_{\mathbf{x}}$ under the action of M_{11}. In the second case, c and d are conjugate under M_{11} to two of $\pm\mathbf{f}$, $\pm\mathbf{g}$, and $\pm(\mathbf{f}+\mathbf{u}) \pm (\mathbf{g}+\mathbf{u})$. Now conjugation by appropriate elements of S_2 and P means that we may assume that $c = \mathbf{f}$ and $d = \mathbf{g}$, $\mathbf{f}+\mathbf{g}$, or $\mathbf{f}-\mathbf{g}$; or that $c = \mathbf{f}-\mathbf{g}$ and $d = \mathbf{f}$. If $c = \mathbf{f}$ and $d = \mathbf{g}$ then conjugation by $x_t x_n x_t$ maps $\infty_{\mathbf{f}} 0_{\mathbf{g}}$ to $\delta_{\mathbf{f},\mathbf{g}}^{-1} \infty_{\mathbf{f}-\mathbf{g}} \infty_{\mathbf{f}+\mathbf{g}} \infty_{¢}^{-1} 0_{¢}^{-1}$, and this is conjugate to $\infty_{\mathbf{f}-\mathbf{g}} 0_{\mathbf{f}+\mathbf{g}-\mathbf{u}}$ by an element of P. A similar argument applies with the other possibilities for c and d. ∎

LEMMA 2.9. *If $t \in P{:}T \setminus P$, then t is N-conjugate to T_1 $T_1\infty_{¢}$, $T_1\delta_{4-7}$, $T_1\infty_{\mathbf{y}}$, or $T_1\infty_{\mathbf{y}+\mathbf{u}}$.*

Proof. Let t be an element of $P{:}T \setminus P$. Conjugation by an element of a Sylow 2-subgroup of M_{11} normalizing T shows that we may assume $t = T_1 \infty_d 0_e \delta$ for some $d, e \in \mathcal{G}$ and $\delta \in P'$. Using the action of T_1 on $\mathcal{G}$, we compute that $t^3 \in P'$ if and only if $\delta_{123}(d) = \delta_{123}(e) = 0$. Now conjugation by ∞_c and 0_c for all $c \in \mathcal{G}$ shows that we may assume that $d, e \in \langle \mathbf{y} \rangle$, and so we may assume $t = T_1 \infty_{\mathbf{y}}^a \delta$ for some $\delta \in P'$ and $a = 0$, 1, or 2. Similarly, we now find that $t^3 \in Z(P)$ if and only if $\delta(\mathbf{f}+\mathbf{g}) = 0$. Now conjugation by ϵ for all $\epsilon \in \bar{\mathcal{G}}^*$ shows that we may assume that $\delta \in \langle \delta_{4-7} \rangle Z(P)$. Hence we may assume $t \in T_1 \infty_{\mathbf{y}}^a \delta_{4-7}^b Z(P)$ for $a, b = 0$, 1, or 2. Conjugation by elements of $X = \langle x_t, x_n \rangle$ allows us to assume $a, b = 0$ or 1. However, $T_1 \infty_{\mathbf{y}} \delta_{4-7}$ does not have order 3, as the action of B shows that $T_1 \infty_{\mathbf{y}} \delta_{4-7}$ is conjugate to $0_{\mathbf{x}} \infty_{\mathbf{y}-\mathbf{f}+\mathbf{g}}$, and this does not have order 3, since $(\mathbf{x}, \mathbf{y}-\mathbf{f}+\mathbf{g}, \mathbf{y}-\mathbf{f}+\mathbf{g}) \neq 0$.
If $a = b = 0$, conjugation by elements of X shows that t is conjugate to T_1 or $T_1\infty_{¢}$. If $a = 0$ and $b = 1$, then conjugation by $\infty_{\mathbf{f}}$ and $0_{\mathbf{f}}$ shows that t is conjugate to $T_1\delta_{4-7}$. If $a = 1$ and $b = 0$, then conjugation by ρ^2 shows that t is conjugate to $T_1\infty_{\mathbf{y}}$ or $T_1\infty_{\mathbf{y}+\mathbf{u}}$. ∎

LEMMA 2.10. *The conjugacy classes of elements of N from the preceding lemmas are*: $\infty_{\mathbf{f}-\mathbf{g}} 0_{\mathbf{f}+\mathbf{g}-\mathbf{u}}$ *is in class* $3B$, $\infty_{\mathbf{y}} 0_{\mathbf{x}}$ *is in class* $3C$, T_1 *is in class* $3A$, $T_1\infty_{¢}$ *is in class* $3A$, $T_1\infty_{\mathbf{y}}$ *is in class* $3C$, $T_1\infty_{\mathbf{y}+\mathbf{u}}$ *is in class* $3C$, *and* $T_1\delta_{4-7}$ *is in class* $3C$.

Proof. First, we note that classes of $6Suz$ fuse to classes in the Monster, depending on their trace on Λ. This fusion is described in Table V. Thus T_1 is in class $3A$, $T_1\infty_{¢}$ is in class $3B$, and $T_1\infty_{\mathbf{y}}$ is conjugate to $T_1 0_{\mathbf{y}}$, which has trace 0 and hence is in class $3C$. By [5, Section 2], $T_1 0_{\mathbf{y}+\mathbf{u}}$ also is in class $3C$. Under the action of B, $\infty_{\mathbf{y}} 0_{\mathbf{x}}$ is conjugate to $\infty_{\mathbf{y}} T_1$, so it is in class $3C$. Also under the action of B, $\infty_{\mathbf{f}-\mathbf{g}} 0_{\mathbf{f}+\mathbf{g}-\mathbf{u}}$ is conjugate to $\delta_{7-4} 0_{\mathbf{f}+\mathbf{g}-\mathbf{u}}$, which is in class $3B$, and $T_1 \delta_{4-7}$ is conjugate to $0_{\mathbf{x}}\infty_{\mathbf{g}-\mathbf{f}}$, and this is conjugate to $0_{\mathbf{x}}\infty_{\mathbf{y}}$, so it is in class $3C$. ∎

2.5. *Subgroups of Type* $3B_4(i)$ *in* $P \cap Q$

The theorems in this section are used in Section 5 to show that certain groups are not conjugate, and in Section 7 to show that certain groups are conjugate.

DEFINITION 2.13. The *type* of an elementary abelian 3-subgroup G is $3A_iB_jC_k$, where G has, respectively, i, j, and k cyclic subgroups with elements from classes $3A$, $3B$, and $3C$.

We omit any classes that have no elements in a subgroup. By Lemma 2.5, the groups $Z(P)$, $\langle \infty_{\mathbf{f}}\delta_{4-7}, \infty_{\mathbf{u}}\rangle$, and $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{u}}\rangle$ all have type $3B_4$. They are not conjugate, however, in $\mathbb{M}$. We follow the notation of [5] and say that their conjugates have type $3B_4(i)$, $3B_4(ii)$, and $3B_4(iii)$, respectively.

We say that an element of $Q \cap P$ lies in the $\mathcal{G}$ layer if it is an element of $Q \cap P \setminus P'$, and a subgroup lies in the $\mathcal{G}$ layer if its nonidentity elements do so. We find representatives of the N-conjugacy classes of subgroups of $Q \cap P$ with type $3B_4(i)$ that lie in the $\mathcal{G}$ layer. Every such subgroup contains elements that correspond to an element of shape $(1^3, 2^3, 0^6)$ of $\mathcal{G}/\langle u\rangle$. There are only 2 N-classes of $3B$ elements of the $\mathcal{G}$ layer that correspond to an element of shape $(1^3, 2^3, 0^6)$: $\infty_{\mathbf{f}+\mathbf{g}}$ and $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$, so every $3B_4(i)$ in the $\mathcal{G}$ layer contains a conjugate of one of these two elements. We find an element g of $6Suz$ that maps $0_{¢}$ to each of these elements. Then each $3B_4(i)$ containing one of these elements is the image of $\langle 0_{¢}, \delta_{ijk}\rangle$ for some i, j, k. The last step is to check which of these are in the $\mathcal{G}$ layer.

THEOREM 2.1. *If H is a group of type $3B_4(i)$ that lies in the $\mathcal{G}$ layer and contains $\infty_{\mathbf{f}+\mathbf{g}}$, then* $H = \langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}\rangle$, *where*

TABLE V
Fusion of Elements of Order 3 of $6Suz$ in $\mathbb{M}$

Trace in $6Suz$	3	3ω	$3\bar{\omega}$	-6	-6ω	$-6\bar{\omega}$	0
Monster class	$3A$	$3B$	$3B$	$3B$	$3A$	$3A$	$3C$

$A, B = 0, 1,$ *or* 2. *Furthermore, each of these groups is N-conjugate to* $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}\delta_{123}\rangle$.

For the proof we will need the following lemma, which can be proved with a computation in $\mathscr{G}^*$.

LEMMA 2.11. *Each δ_{1ij} with $4 \le i \le 9$ and $10 \le j \le 12$ can be represented as* $\delta_{1ij} = \delta_{4-7}^{\pm 1}\delta_{2-3}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$.

Proof of Theorem 2.1. The element $B\sigma_1 B^{-1}$ maps $\infty_{\mathbf{u}}$ to $\infty_{\mathbf{f}+\mathbf{g}}$, and the groups of type $3B_4(i)$ that contain $0_{¢}$ correspond to the long cocode words in $\bar{\mathscr{G}}^*$. Suppose that δ is a long cocode word. Then either δ or $-\delta$ is equal to δ_{1ij} for some $1 < i < j$. Now $B\sigma_1 B^{-1}$ maps δ_{1ij} to an element of $P \cap Q \setminus P'$ if and only if $4 \le i \le 9$ and $10 \le j \le 12$; $B\sigma_1 B^{-1}$ maps δ_{123} to δ_{123}, and δ_{1ij} to an element of $Q \setminus P$ if $j < 10$ or $i < 4$ or $i > 9$.

Now $B\sigma_1 B^{-1}$ maps δ_{4-7} to $\infty_{\mathbf{x}-\mathbf{y}}$, δ_{2-3} to $\infty_{\mathbf{f}-\mathbf{g}}$, and fixes δ_{11-12} and δ_{123}. Using Lemma 2.11, every group of type $3B_4(i)$ in the $\mathscr{G}$-layer that contains $\infty_{\mathbf{f}+\mathbf{g}}$ contains an element of the form $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$. The permutation ρ^2 acts by fixing $\infty_{\mathbf{f}+\mathbf{g}}$ and switching the elements $\infty_{\mathbf{x}-\mathbf{y}}\infty_{\mathbf{f}-\mathbf{g}}^{-A} \cdot \delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$ and $\infty_{\mathbf{x}-\mathbf{y}}^{-1}\infty_{\mathbf{f}-\mathbf{g}}^{A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$. The permutation T_1 centralizes $\infty_{\mathbf{f}+\mathbf{g}}$ and maps $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$ to $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp(A+1)}\delta_{11-12}^{A+B} \cdot \delta_{123}^{A^2-B^2-A-B-1}$; observe that $A^2-B^2-A-B-1 = (A+1)^2-(B-1)^2-1$. The action of $0_{\mathbf{f}-\mathbf{g}}$ is a little bit more subtle; it fixes $\infty_{\mathbf{f}+\mathbf{g}}$ and maps the element $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$ to the element $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A} \cdot \delta_{11-12}^{A+B\mp 1}\delta_{123}^{A^2-B^2\pm A-1}$. Observe that $A^2-B^2\pm A-1 = A^2-(B\mp 1)^2-1$ when $\pm A = \mp B - 1$; that is, when $B = \mp 1 - A$, and $-A^2+B^2\mp A+1 = A^2-(A-B\pm 1)^2-1$ when $A^2-AB+B^2\mp(A+B) = 0$; that is, when $B = -A$ or $B = \pm 1 - A$.

Combining the action of ρ^2, T_1, and $0_{\mathbf{f}-\mathbf{g}}$ shows that every group of type $3B_4(i)$ in the $\mathscr{G}$ layer that contains $\infty_{\mathbf{f}+\mathbf{g}}$ is N-conjugate to $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}\delta_{123}\rangle$. ∎

THEOREM 2.2. *Every subgroup $H < P \cap Q$ of type $3B_4(i)$ that lies in the $\mathscr{G}$ layer and contains $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$ is N-conjugate to the subgroup $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}\delta_{123}\rangle$ or the subgroup $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$. Furthermore, the stabilizer in N of $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$ fixes the subgroup $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$.*

Proof. This is similar to the proof of Theorem 2.1. Conjugation by $B\pi B\sigma_1 B^{-1}$ maps $\infty_{\mathbf{u}}$ to $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$, and we can determine which elements of $\bar{\mathscr{G}}^*$ are mapped to the $\mathscr{G}$ layer by conjugation by this element. This gives 19 subgroups of $P \cap Q$ of type $3B_4(i)$ that contain $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$. Each is generated by $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$ and an element $\infty_c\delta_c$ for some $\delta_c \in \bar{\mathscr{G}}^*$, and the code words c are distinct. For the moment we ignore the terms δ_c. The code words c corresponding to these groups are $\mathbf{f}+\mathbf{g}$, $-\mathbf{z}\pm\mathbf{x}+\mathbf{u}$, $\mathbf{z}\pm\mathbf{y}+\mathbf{f}-\mathbf{u}$,

$\mathbf{z} \pm (\mathbf{x} + \mathbf{y}) + \mathbf{g} - \mathbf{u}$, $\mathbf{z} + \mathbf{f}$, $\mathbf{z} + \mathbf{g}$, $-\mathbf{z} \pm (\mathbf{f} - \mathbf{g}) + \mathbf{u}$, $\mathbf{z} \pm (\mathbf{f} + \mathbf{g} - \mathbf{u})$, $\mathbf{f} + \mathbf{g} \pm \mathbf{x} - \mathbf{u}$, $-\mathbf{f} \pm (\mathbf{x} + \mathbf{y}) + \mathbf{u}$, and $-\mathbf{g} \pm \mathbf{y} + \mathbf{u}$. The stabilizer of $\langle \mathbf{x} - \mathbf{y} \rangle$ in M_{11} contains $T_1^{\sigma_1}$, ξ, and σ_2, and it fixes $\mathbf{z} + \mathbf{f} + \mathbf{g} - \mathbf{u}$ and acts transitively on the other 18 code words. Note that the stabilizer of $\mathbf{x} - \mathbf{y}$ is the conjugate by σ_1 of the stabilizer of $\mathbf{f} + \mathbf{g}$. ∎

3. A CONTAINS A SUBGROUP OF TYPE $3B_4(i)$

Recall that S is the Sylow 3-subgroup of $\mathbb{M}$ defined in Definition 2.9, and that S is equal to $PT\langle x_t \rangle$.

LEMMA 3.1. *A maximal elementary abelian subgroup A of the Monster is conjugate to a subgroup of $P.T$ or $Q.T$.*

Proof. Suppose that A is a maximal elementary abelian 3-subgroup of $\mathbb{M}$ contained in S, and A is not contained in subgroup of $P.T$ or $Q.T$. Then we may assume it contains some element $\alpha = x_t \delta 0_c \infty_d t$, where $\delta \in P'$, $c, d \in \mathcal{G}$, and $t \in T$. The factor x_t is needed so that α is not in $P.T$, and the factor 0_c is needed so that α is not in $Q.T$. Conjugation by 0_d shows that we may assume d is the identity, so we may assume $\alpha = x_t \delta 0_c t$. If t is the identity, a computation using Eq. (1) shows that $(x_t \delta 0_c)^3 = \delta_{c,c} \infty_{¢}^{c_1 - \delta(c) - \Psi(c)}$. Thus if α has order 3, then c has shape $\pm(1^6, 0^6)$ or $(1^6, 2^6)$.

After conjugation by elements of S_2 and T, we may assume that $c \in \pm\mathbf{f} + \langle \mathbf{u} \rangle$ or $c \in \pm\mathbf{z} + \langle \mathbf{u} \rangle$. If $c \in \pm\mathbf{f} + \langle \mathbf{u} \rangle$, then $C_S(\alpha) < \langle P', \infty_c, T, \alpha \mid c \in \langle \mathbf{f}, \mathbf{g}, \mathbf{x}, \mathbf{y} \rangle \rangle$. In this case, both α^B and $C_S(\alpha)^B$ are contained in $P{:}\,T$. (Note that T_1^B and T_2^B have the same action on Q/Q' as $0_{\mathbf{x}}$ and $0_{\mathbf{x}-\mathbf{y}}$, respectively, and from the structure of R we see that they are in P.) In the case that $c \in \pm\mathbf{z} + \langle \mathbf{u} \rangle$, then $C_S(\alpha) < \langle P', \infty_{\mathbf{z}}, \alpha \rangle$. Although the element $\pi \in M_{11}$, which is listed in Table II, does not normalize the Sylow subgroup S, it does satisfy $\mathbf{z}^\pi = \mathbf{u} - \mathbf{f}$. Thus $\alpha^{\pi B}$ and $C_S(\alpha)^{\pi B}$ are in P.

Thus we are left with the possibility that t is not the identity. Since $N_{M_{11}}(T)$ is transitive on the nonidentity elements of T, we may assume that $t = T_1$. We compute using Eq. (1) that

$$\alpha^3 = \delta \delta^{T_1^2} \delta^{T_1} \delta_{c,c} \delta_{c,c}^{T_1^2} \delta_{c,c^{T_1^2}} \infty_c^{-1} \infty_c^{T_1^2} 0_c 0_c^{T_1^2} 0_c^{T_1} \infty_{¢}^{c_1 - \delta^{T_1}(c) - \Psi(c)} 0_{¢}^{\delta^{T_1}(c) - \delta(c^{T_1})}. \quad (3)$$

If α has order 3, then $c^{T_1} = c$, so $c \in \langle \mathbf{f}, \mathbf{g}, \mathbf{u} \rangle$, and for these values of c we have

$$\alpha^3 = \delta \delta^{T_1^2} \delta^{T_1} \infty_{¢}^{c_1 - \delta(c) - \Psi(c)}. \quad (4)$$

Since $\delta \delta^{T_1^2} \delta^{T_1} = 0$, we have $\delta \in \langle \delta_{123}, \delta_{2-3}, \delta_{11-12}, \delta_{4-7} \rangle$.

Using Eq. (1) we find

$$C_S(\alpha) < \left\langle \delta, \infty_d, 0_e, T_1, T_2, \alpha \mid \delta \in \bar{\mathcal{G}}^*, \delta_{123}(d) = 0, e \in \langle \mathbf{f}, \mathbf{g}, \mathbf{u} \rangle \right\rangle. \quad (5)$$

From [3] we have that 0_c acts on Q/Q' as the diagonal matrix $diag(\omega^{c_i})$ acts on $\Lambda_{\mathbb{C}}/\Theta\Lambda_{\mathbb{C}}$. This fact, along with Table III, shows that $\delta^B \in \langle \mathbf{f} - \mathbf{g} \rangle P'$, that $\infty_d^B \in P \cap Q$ when $\delta_{123}(d) = 0$, that $0_d^B = 0_d$ when $d \in \langle \mathbf{f}, \mathbf{g}, \mathbf{u} \rangle$, and $\alpha^B = \infty_{\tilde{\mathbf{x}}}\delta^B 0_c 0_{\mathbf{x}}$. Since $\delta^B \in P \cap Q$, we see that α^B and $C_S(\alpha)^B$ are both contained in $P{:}\,T$. ∎

THEOREM 3.1. *Every maximal elementary abelian subgroup A of the Monster is conjugate to a subgroup of $P.T$.*

Proof. Suppose that A is contained in $Q.T$. Since the largest $3A$–pure subgroup of the Monster has order 9, if the order of $A \cap Q$ is at least 81, it follows that A contains a conjugate of $Z(P)$ and so is conjugate to a subgroup of $P.T$. If A is contained in Q, then A has order 3^7 and so is conjugate to a subgroup of P.

If $[A{:}\, A \cap Q] = 3$, we may assume $T_1 q \in A$, where $q \in Q$. From [5, Section 4] we may assume that q is one of the identity, $0_{¢}$, or δ_{4-7}. Now $A \cap Q$ is conjugate to a subgroup of $C_Q(T_1)$, and $C_Q(T_1) = \langle \infty_{¢}, 0_{¢}, \delta_{123}, \delta_{2-3}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, x_t \rangle$. Since δ_{123} centralizes $A \cap Q$ and $T_1 q$, it must be contained in A, and as it is in class $3B$, A contains a subgroup conjugate to $Z(P)$, as required.

If $[A{:}\, A \cap Q] = 9$, then $A \cap Q \subset C_Q(T)$ and $C_Q(T) = \langle \infty_{¢}, 0_{¢}, \delta_{123}, \infty_f, \infty_g \rangle$. Now suppose that $T_1 q_1$, $T_2 q_2 \in A$, where $q_i \in Q$. The requirement that $T_1 q_1$ and $T_2 q_2$ commute implies that

$$q_1^{T_2} q_2 = q_2^{T_1} q_1. \tag{6}$$

We can assume, as above, that q_1 is one of the identity, $0_{¢}$, or δ_{4-7}. Equation (6) implies that q_2 must lie in $C_Q(T_1)$ in the first two cases, or that q_2 lies in $C_Q(T_1)P'$ in the last case (i.e., $q_2 = \delta c$, where $\delta^{T_1}\delta_{4-7} = \delta_{4-7}^{T_2}\delta$ and $c \in C_Q(T_1)$.) As before, δ_{123} centralizes $T_1 q_1$, $T_2 q_2$, and $A \cap Q$, so it is contained in A. Thus A contains a subgroup conjugate to $Z(P)$. ∎

4. GROUPS CONTAINING $Z(R)$

We have shown that every maximal elementary abelian subgroup is conjugate to a subgroup of $P.T$. Our approach is first to classify the conjugacy classes of maximal elementary subgroups of $P{:}\,T$ under the action of N, and then to determine their fusion in $\mathbb{M}$. In determining the N-classes, we divide the problem into three cases: $[A{:}\, A \cap P] = 9$, $[A{:}\, A \cap P] = 3$, or $[A{:}\, A \cap P] = 1$. In the cases $[A{:}\, A \cap P] = 9$ or 3, it turns out that every maximal elementary abelian subgroup contains $Z(R)$, so we further divide the case $[A\colon A\cap P]=1$ into the subcases: A contains $Z(R)$, or A does not contain an N-conjugate of $Z(R)$.

First, we record the following lemma, which we invoke for many of the cases considered in this section.

LEMMA 4.1. *Every maximal elementary abelian subgroup of* $Z(R)\langle\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$ *is N-conjugate to one of the groups* $D_1=Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}}\rangle$, $D_2=Z(R)\langle\infty_{\mathbf{f}},0_{\mathbf{f}}\rangle$, $D_3=Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}\rangle$, $D_4=Z(R)\langle\infty_{\mathbf{f}-\mathbf{g}},0_{\mathbf{f}+\mathbf{g}}\rangle$, *or* $D_5=Z(R)\langle\infty_{\mathbf{f}}0_{\mathbf{g}},\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}\rangle$.

Proof. First we observe that $Z(R)$ is the center of $Z(R)\langle\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$. Suppose that D contains an element of the form ∞_c where c has shape $(1^6,0^6)$. After conjugation by χ, if necessary, we may assume that $\infty_{\mathbf{f}}\in D$. Then D is contained in $Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{f}}\rangle$, so $D=Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}}\rangle$, $D=Z(R)\langle\infty_{\mathbf{f}},0_{\mathbf{f}}\rangle$, $D=Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}\rangle$, or $D=Z(R)\langle\infty_{\mathbf{f}},\infty_{\mathbf{g}}0_{\mathbf{f}}\rangle$. Conjugation by x_d shows that the last two are conjugate, so $D=D_1$, D_2, or D_3.

Suppose that D contains an element of the form ∞_c where c has shape $(1^3,2^3,0^6)$. After conjugation by ρ, if necessary, we may assume that $\infty_{\mathbf{f}+\mathbf{g}}\in D$. Then D is contained in $\langle\infty_{\mathbf{f}+\mathbf{g}},\infty_{\mathbf{f}-\mathbf{g}},0_{\mathbf{f}-\mathbf{g}}\rangle$ and so is N-conjugate to $\langle\infty_{\mathbf{f}+\mathbf{g}},0_{\mathbf{f}-\mathbf{g}}\rangle$ or $D=Z(R)\langle\infty_{\mathbf{f}+\mathbf{g}},\infty_{\mathbf{f}-\mathbf{g}}\rangle$. The former is D_4, while the latter is D_1.

Finally, suppose that D does not contain an element of the form i_c. By Lemma 2.8, we may assume that D contains $\infty_{\mathbf{f}}0_{\mathbf{g}}$. This implies that D is contained in $D_5=Z(R)\langle\infty_{\mathbf{f}}0_{\mathbf{g}},\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}\rangle$; hence $D=D_5$. ∎

LEMMA 4.2. *If A is a maximal elementary abelian subgroup of $P\colon T$ with* $[A\colon A\cap P]=9$, *then* $A=Z(R)D_iT$, *where D_i is as given in Lemma* 4.1.

Proof. Since T acts trivially on AP'/P', and on $(A\cap P')/Z(P)$, it follows that $A\cap P\subset Z(P)\langle\delta_{123},\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$. Furthermore, since δ_{123} commutes with all of the generators listed above, as well as with the possible elements of $A\setminus P$, we may assume that $Z(R)<A$.

Now A contains elements T_1p_1 and T_2p_2 with p_1 and p_2 elements of P. By Lemma 2.10, we may assume that p_1 is either 1, $\infty_{\mathbf{y}}$, or δ_{4-7}. Since T_2 does not commute with $\infty_{\mathbf{y}}$ or δ_{4-7}, we may assume that $T_1\in A$. Now p_2 must commute with T_1, so we have $p_2\in\langle\delta_{123},\delta_{2-3},\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$. Since T_2 centralizes $\langle\delta_{123},\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$, p_2 must centralize $A\cap\langle\delta_{123}\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$. Clearly p_2 centralizes $Z(P)$ and T_2p_2, so by maximality of A, we have $p_2\in A$. Then we may assume that $T<A$. Thus $A=Z(R)D_iT$, where D_i is a maximal elementary abelian subgroup of $\langle\infty_{\mathbf{f}},0_{\mathbf{f}},\infty_{\mathbf{g}},0_{\mathbf{g}}\rangle$. ∎

LEMMA 4.3. *If A is a maximal elementary abelian subgroup of $P\colon T$ with* $[A\colon A\cap P]=3$, *then A is conjugate to a group of the form* $Z(R)D_i\langle\delta_{2-3},T_1\rangle$, *where* D_i *is as given in Lemma* 4.1, *or* A *is conjugate to the group* $B_6 = Z(R)\langle \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, T_1\infty_{\mathbf{y}} \rangle$ *or the group* $B_7 = Z(R)\langle \infty_{\mathbf{f+g}}, \delta_{2-3}, T_1\delta_{4-7} \rangle$.

Proof. Similar to the previous lemma, we find that

$$A \cap P \subset Z(P)\langle \delta_{123}, \delta_{2-3}, \infty_{\mathbf{f}}, 0_{\mathbf{f}}, \infty_{\mathbf{g}}, 0_{\mathbf{g}} \rangle.$$

As in the previous lemma, we may assume that $Z(R) < A$.

Now by Lemma 2.10, any element $T_1 p_1$ in A is conjugate to T_1, $T_1\infty_{\mathbf{y}}$, or $T_1\delta_{4-7}$, and the conjugacy is realized by an element of N that normalizes $P{:}\,T$, so we may assume that one of T_1, $T_1\infty_{\mathbf{y}}$, or $T_1\delta_{4-7}$ is an element of A.

If $T_1 \in A$, then $\delta_{2-3} \in A$, since it commutes with all possible elements of A. Thus $A = Z(R)\langle \delta_{2-3}, T_1 \rangle D_i$, where D_i is a maximal elementary abelian subgroup of $\langle \infty_{\mathbf{f}}, 0_{\mathbf{f}}, \infty_{\mathbf{g}}, 0_{\mathbf{g}} \rangle$.

If $T_1\infty_y \in A$, then $A \cap P < C_P(T_1\infty_{\mathbf{y}}) = Z(P)\langle \delta_{123}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}} \rangle$. But the group on the right side of this inequality is elementary abelian, so we must have $A = B_6 = Z(P)\langle \delta_{123}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, T_1\infty_{\mathbf{y}} \rangle$.

If $T_1\delta_{4-7} \in A$, then we have $A \cap P < Z(P)\langle \delta_{123}, \delta_{2-3}, \infty_{\mathbf{f+g}}, 0_{\mathbf{f+g}} \rangle$. Now δ_{123} and δ_{2-3} are in the center of the group on the right hand side. Furthermore, $\infty_{\mathbf{f+g}}$ and $0_{\mathbf{f+g}}$ do not commute. Using Eq. (1), we can verify that the groups $Z(P)\langle \delta_{123}, \delta_{2-3}, i_{\mathbf{f+g}} \rangle$ for $i = \infty$, 0, 1, and 2 are conjugate under the action of $\langle x_t, x_n x_d \rangle$, which also fixes $T_1\delta_{4-7}$. Thus A is N conjugate to $B_7 = Z(P)\langle \delta_{123}, \delta_{2-3}, \infty_{\mathbf{f+g}}, T_1\delta_{4-7} \rangle$. ∎

LEMMA 4.4. *If* A *is a maximal elementary abelian subgroup of* P *with* $Z(R) < A$, *then* $A = Z(R')D_i$, *where* D_i *is as given in Lemma* 4.1, *or else* A *is conjugate to one of the groups listed in Lemmas* 4.2 *and* 4.3.

Proof. Suppose that $A < P$ and $Z(R) < A$. This implies that $A < R$, and so if A is not contained in R', then it is conjugate in $N_{\mathbb{M}}(R)$ to a group A_0 with $[A_0{:}\, A_0 \cap P] = 3$ or 9, and thus is conjugate to a group described in one of the previous two lemmas. Thus we only need to consider groups $A < R'$, and so we may assume that $Z_2(R) = Z(R')$ is contained in A.

Thus $A = Z(R')D_i$, where D_i is a maximal elementary abelian subgroup of $\langle \delta_{4-7}, \delta_{147}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, 0_{\mathbf{f}}, 0_{\mathbf{g}} \rangle$, and the action of $N_{\mathbb{M}}(R)$ shows that A is conjugate to $A = Z(R')D_i$, where D_i is as given in Lemma 4.1. ∎

THEOREM 4.1. *Every maximal elementary abelian* 3-*subgroup of* $P{:}\,T$ *that contains* $Z(R)$ *is* N-*conjugate to one of*

$$A_i = Z(R)D_i\langle T_1, T_2 \rangle$$

$$B_i = Z(R)D_i\langle \delta_{2-3}T_1 \rangle$$

$$B_6 = Z(R)\langle \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, T_1\infty_y \rangle$$
$$B_7 = Z(R)\langle \delta_{2-3}, \infty_{\mathbf{f}+\mathbf{g}}, T_1\delta_{4-7} \rangle$$
$$C_i = Z(R')D_i,$$

where D_i *is as given in Lemma* 4.1.

LEMMA 4.5. *In the notation of Theorem* 4.1, A_1 *is conjugate to* B_4, B_1 *is conjugate to* C_4, A_5 *is conjugate to* C_5, *and* B_6 *is conjugate to* B_7.

Proof. The group B_1 is conjugate to C_4 by $B\sigma_2B^{-1}$, the group A_1 is conjugate to B_4 by $B\sigma_1B^{-1}$. Conjugation by $0_{\mathbf{f}+\mathbf{g}}B\sigma_1B^{-1}$ shows that A_5 fuses with B_5, while conjugation by $B(T_1^{-1})^{\sigma_2}$ shows that B_6 fuses with B_7. ∎

LEMMA 4.6. *The types of the maximal elementary abelian* 3*-subgroups of* $P{:}T$ *that contain* $Z(R)$ *are as given in Table* VI.

Proof. First we consider the groups A_i. By Lemma 2.10, for each non-identity element $t \in T$, the coset $Z(R)D_it$ has 27 elements in class $3A$ and 216 elements in class $3B$, so in total these cosets give us 108 cyclic subgroups with elements from class $3A$ and 864 cyclic subgroups with elements from class $3B$. By Lemma 2.5, $Z(R)\infty_{\mathbf{f}}$ and $Z(R)\infty_{\mathbf{g}}$ give 36 $3A$ subgroups and 18 $3B$ subgroups; $Z(R)\infty_{\mathbf{f}+\mathbf{g}}$ and $Z(R)\infty_{\mathbf{f}-\mathbf{g}}$ give 18 $3A$ subgroups and 36 $3B$ subgroups; and $Z(R)$ has 13 $3B$ subgroups, so in total $Z(R)D_1$ has 54 $3A$ subgroups and 67 $3B$ subgroups. Similarly, $Z(R)D_2$ has 72 $3A$ subgroups and 49 $3B$ subgroups; $Z(R)D_3$ and $Z(R)D_4$ each have 18 $3A$ subgroups and 103 $3B$ subgroups; and $Z(R)D_5$ has 121 $3B$ subgroups.

Second we consider the groups B_i. By Lemma 2.10, the coset $Z(R)D_iT_1$ has 81 elements in class $3A$ and 648 elements in class $3B$. By Lemma 2.5, $Z(R)\langle\delta_{2-3}\rangle D_1$ has 189 $3A$ subgroups and 175 $3B$ subgroups; $Z(R)\langle\delta_{2-3}\rangle D_2$ has 243 $3A$ subgroups and 121 $3B$ subgroups; $Z(R)\langle\delta_{2-3}\rangle D_3$ has 81 $3A$ subgroups and 283 $3B$ subgroups; and $Z(R)\langle\delta_{2-3}\rangle D_5$ has 27 $3A$ subgroups and 337 $3B$ subgroups. Furthermore, by Lemma 2.10 every element of $B_6 \setminus Z(R)D_1$ is in class $3C$, so B_6 has 54 $3A$ subgroups, 67 $3B$ subgroups, and 243 $3C$ subgroups.

TABLE VI
Types of Maximal Elementary Abelian 3-subgroups of $P{:}T$.

Group	A_1	A_2	A_3	A_4	A_5
Type	$3A_{162}B_{931}$	$3A_{180}B_{913}$	$3A_{126}B_{967}$	$3A_{126}B_{967}$	$3A_{108}B_{985}$
Group	B_1	B_2	B_3	B_5	B_6
Type	$3A_{270}B_{823}$	$3A_{324}B_{769}$	$3A_{162}B_{931}$	$3A_{108}B_{985}$	$3A_{54}B_{67}C_{243}$
Group	C_1	C_2	C_3		
Type	$3A_{594}B_{499}$	$3A_{2214}B_{1066}$	$3A_{270}B_{823}$		

Third we consider the groups C_i. By Lemma 2.5, C_1 has 594 $3A$ subgroups and 499 $3B$ subgroups; C_2 has 2214 $3A$ subgroups and 1066 $3B$ subgroups; and C_3 has 270 $3A$ subgroups and 823 $3B$ subgroups. ∎

5. NONCONJUGACY OF A_3-A_4, A_1-B_3, A_5-B_5, B_1-C_3

LEMMA 5.1. *Suppose that G is a subgroup of Q of type $3B_4(i)$ with $|G \cap P'| = 3$ and $G \cap Z(P) = 1$. Then $G = \langle \delta_{ijk}\zeta, \infty_c\delta \rangle$, where $\zeta \in Z(P)$, $\delta(\tilde{c}) = 0$, and the support of c is the union of the supports of representatives of δ_{ijk}.*

Proof. Every group of type $3B_4(i)$ in Q that contains $0_¢$ but not $\infty_¢$ has the form $\langle 0_¢, \delta_{ijk}\infty_¢^{\epsilon} \rangle$, where $1 \le i < j < k \le 12$ and $\epsilon = 0$, 1, or 2. Conjugation of $0_¢$ by B shows that every group of type $3B_4(i)$ in Q that contains δ_{123} but no element $i_¢$ for $i = \infty$, 0, 1, or 2 contains an element of the form $\infty_c\delta$, where $c \in \langle \mathbf{f}, \mathbf{g} \rangle$, $\delta \in P'$, and $\delta(\mathbf{f}) = \delta(\mathbf{g}) = 0$. Since $\tilde{c}$ is also in $\langle \mathbf{f}, \mathbf{g} \rangle$ when c is in $\langle \mathbf{f}, \mathbf{g} \rangle$, the lemma is true when $\delta_{ijk}\zeta = \delta_{123}$. Now conjugation by appropriate powers of $\infty_{\mathbf{z}}$ and $0_{\mathbf{z}}$ maps δ_{123} to $\delta_{123}\zeta$ for any $\zeta \in Z(P)$, and M_{11} is 3-transitive, so the lemma is proved. ∎

5.1. *Subgroups of Type* $3B_4(i)$ *in* A_3 *and* A_4

LEMMA 5.2. *Let $A_3^* = A_3^{Bx_n}$. The group A_3^* does not have any subgroups G of type $3B_4(i)$ with $|G \cap P'| = 3$ and $G \cap Z(P) = 1$.*

Proof. It follows from Eq. (1) and Table III, that

$$A_3^* = \langle \infty_¢, 0_¢, \delta_{123}, \delta_{7-10}, \infty_{\mathbf{f}}\delta_{4-10}, \infty_{\mathbf{x}}, \infty_{\mathbf{y}} \rangle.$$

Suppose that G is a subgroup of A_3^* with type $3B_4(i)$, $|G \cap P'| = 3$, and $G \cap Z(P) = 1$. Then G contains either $\delta_{123}\zeta$ or $\delta_{234}\zeta$, for some $\zeta \in Z(P)$. First suppose that G contains $\delta_{123}\zeta$. By Lemma 5.1, G contains an element of the form $\infty_c\delta$ where $c \in \langle \mathbf{f}, \mathbf{g}, \mathbf{u} \rangle \setminus \langle \mathbf{u} \rangle$, $\delta \in P'$, and $\delta(\tilde{c}) = 0$. We may assume that $c = \mathbf{f} + \alpha\mathbf{u}$ for some $\alpha \in \mathbb{F}_3$, so $\delta_c = \delta_{10-4} + \beta\delta_{7-10} + \gamma\delta_{123}$ for β, $\gamma \in \mathbb{F}_3$. But this implies that $\delta_c(\tilde{c}) \neq 0$, a contradiction. A similar argument works if G contains $\delta_{234}\zeta$, so A_3^* does not contain any such subgroup G. ∎

LEMMA 5.3. *Let $A_4^* = A_4^{Bx_n}$. The group A_4^* has 54 subgroups G of type $3B_4(i)$ with $|G \cap P' = 3|$ and $G \cap Z(P) = 1$.*

Proof. As in Lemma 5.2, we get

$$A_4^* = \langle \infty_¢, 0_¢, \delta_{123}, \delta_{4-7}, \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}}, \infty_{\mathbf{y}} \rangle.$$

Suppose that G is a subgroup of A_4^* with type $3B_4(i)$, $|G \cap P'| = 3$, and $G \cap Z(P) = 1$. Then G contains either $\delta_{123}\zeta$, $\delta_{489}\zeta$, or $\delta_{567}\zeta$, for some

$\zeta \in Z(P)$. Suppose that G contains $\delta_{123}\zeta$. By Lemma 5.1, G contains an element of the form $\infty_c \delta \infty_{¢}$, where $c \in \langle \mathbf{f}+\mathbf{g}, \mathbf{u}\rangle \setminus \langle \mathbf{u}\rangle$ and $\delta(\tilde{c}) = 0$. Since $\delta \in A_4^*$, this implies that $\delta \in \langle \delta_{123}\rangle$, so we may assume $\delta = 0$. Examining the effect of conjugation by $0_{\mathbf{z}}$ shows that $\zeta \in \langle \infty_{¢}\rangle$. Thus we have shown that $G = \langle \delta_{123}\infty_{¢}^{\alpha}, \infty_{\mathbf{f}+\mathbf{g}}\infty_{\mathbf{u}}^{\beta}\infty_{¢}^{\gamma}\rangle$ for $\alpha, \gamma \in \mathbb{F}_3$ and $\beta = 0$ or 1.

The permutation η shows that the stabilizer in M_{11} of A_4^* is transitive on $\{\langle \delta_{123}, \delta_{489}, \delta_{567}\rangle\}$. Thus A_4^* has a total of 54 subgroups G of type $3B_4(i)$ with $|G \cap P' = 3|$ and $G \cap Z(P) = 1$. ∎

The steps for counting subgroups of type $3B_4(i)$ in A_3 and A_4 whose intersection with P' is trivial are

1. Find the orbits of subcodes corresponding to a subgroup of type $3B_4(i)$.
2. Find a permutation that maps $\langle \mathbf{f}+\mathbf{g}, \mathbf{x}-\mathbf{y}\rangle$ to the subcode.
3. Find elements 0_c whose composition with the permutation maps $\infty_{\mathbf{f}+\mathbf{g}}$ to possible elements of the subgroup.
4. Determine the images of $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp A}\delta_{11-12}^{A+B}\delta_{123}^{A^2-B^2-1}$ under the action of the permutation and 0_c.

LEMMA 5.4. *Let $A_3^* = A_3^{Bx_n}$. The group A_3^* has* 144 *subgroups G of type $3B_4(i)$ with $G \cap P' = 1$.*

Proof. Let G be a subgroup of A_3^* of type $3B_4(i)$ with $G \cap P' = 1$. Then G lies in the $\mathcal{G}$ layer and corresponds to a subcode of $\mathcal{G}$. By Theorems 2.1 and 2.2, G is N-conjugate to $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}\rangle$ or $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$. If G is N-conjugate to $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$, its subcode has two code words of shape $(1^6, 0^6)$, whose supports intersect in a set of size 3. If it is N-conjugate to $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}\rangle$, its subcode has a word of weight 9, and three words of shape $(1^3, 2^3, 0^6)$ whose support is contained in the support of the word of weight 9. Since $\mathbf{f}$ and $\mathbf{u}-\mathbf{f}$ are the only code words of shape $(1^6, 0^6)$ corresponding to elements of A_3^*, the subcode must be a conjugate of $\langle \mathbf{f}+\mathbf{g}, \mathbf{x}-\mathbf{y}\rangle$. Now A_3^* is stabilized by the subgroup $M_3 = \langle \rho, \rho^{\chi}\rangle$ of M_{11}. The subcode $\langle \mathbf{f}, \mathbf{x}, \mathbf{y}, \mathbf{u}\rangle$ has three orbits of words of shape $(1^6, 2^3, 0^3)$ under the action of M_3, with representatives $\mathbf{f}+\mathbf{x}$, $\mathbf{f}+\mathbf{x}-\mathbf{y}+\mathbf{u}$, and $\mathbf{x}+\mathbf{u}$. Furthermore, there are two orbits of words of shape $(1^3, 2^3, 0^6)$ in $\langle \mathbf{f}, \mathbf{x}, \mathbf{y}, \mathbf{u}\rangle$, with representatives $\mathbf{x}$ and $\mathbf{f}+\mathbf{x}-\mathbf{u}$. By examining the supports of these representatives, we see that any subcode of $\langle \mathbf{f}, \mathbf{x}, \mathbf{y}, \mathbf{u}\rangle$ that is M_{11}-conjugate to $\langle \mathbf{f}+\mathbf{g}, \mathbf{x}-\mathbf{y}\rangle$ is M_3-conjugate to $\langle \mathbf{f}+\mathbf{x}, \mathbf{y}\rangle$.

Now we may assume that G is a subgroup of A_3^* with type $3B_4(i)$, which contains $\infty_{\mathbf{f}+\mathbf{x}}\delta$. Since $\delta(\widetilde{\mathbf{f}+\mathbf{x}}) = 0$, we may write $\delta = \delta_{4-10}\delta_{7-10}^{A}\delta_{123}$ for $A = 0$, 1, or 2.

Let γ be the permutation $\sigma_2 T_1 \sigma_1 T_2$. Then γ maps $\mathbf{f}+\mathbf{g}$ to $\mathbf{f}+\mathbf{x}$, $\mathbf{x}-\mathbf{y}$ to $\mathbf{f}+\mathbf{x}-\mathbf{y}$, $\mathbf{f}-\mathbf{g}$ to $-\mathbf{z}-\mathbf{g}+\mathbf{y}+\mathbf{u}$, δ_{11-12} to δ_{6-7}, and δ_{123} to $\delta_{5,9,11}$. Let $c = -\mathbf{f}-\mathbf{g}-\mathbf{x}-\mathbf{y}+A(-\mathbf{f}+\mathbf{g}+\mathbf{y})+E(\mathbf{z}+\mathbf{g}-\mathbf{y})$. Then conjugation by 0_c maps $\infty_{\mathbf{f}+\mathbf{x}}$ to $\infty_{\mathbf{f}+\mathbf{x}}\delta_{4-10}\delta_{7-10}^{A}\delta_{123}$, and it maps $\infty_{\mathbf{f}+\mathbf{x}-\mathbf{y}}$ to $\infty_{\mathbf{f}+\mathbf{x}-\mathbf{y}}\delta_{5-1}\delta_{1-8}^{A}\delta_{7-6}^{E}$. Since G is the conjugate of $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\infty_{\mathbf{f}-\mathbf{g}}^{\mp F}\delta_{11-12}^{F+B}\delta_{123}^{F^2-B^2-1}\rangle$ by $\gamma 0_c$ for some F and B, and G lies in A_3*, it follows that we may assume that

$$G = \left\langle \infty_{\mathbf{f}+\mathbf{x}}\delta_{4-10}\delta_{7-10}^{A}\delta_{123}, \infty_{\mathbf{f}+\mathbf{x}-\mathbf{y}}^{\pm 1}\delta_{6-7}^{B-E}\delta_{5,9,11}^{-B^2-1}\delta_{5-1}^{\pm 1}\delta_{1-8}^{\pm A}\right\rangle$$

for some A, B, and E equal to 0, 1, or 2.

Since the second generator is in A_3^*, we have

$$\delta_{4-10}^{\pm 1}\delta_{7-10}^{C}\delta_{123}^{D} = \delta_{6-7}^{B\mp E}\delta_{5,9,11}^{-B^2-1}\delta_{5-1}^{\pm 1}\delta_{1-8}^{\pm A} \tag{7}$$

for some C and D equal to 0, 1, or 2. Examining the value of each side of Eq. (7) on elements of $\mathscr{G}$ shows that $A = 0$, $C = B \mp E = B^2 + 1 \pm 1$, and $D = 0$. Thus the only possibilities for G are $\langle \infty_{\mathbf{f}+\mathbf{x}}\delta_{4-10}\delta_{123}, \infty_{\mathbf{f}+\mathbf{x}-\mathbf{y}}\delta_{4-10}\delta_{7-10}^{-1}\rangle$ and $\langle \infty_{\mathbf{f}+\mathbf{x}}\delta_{4-10}\delta_{123}, \infty_{\mathbf{f}+\mathbf{x}-\mathbf{y}}\delta_{4-10}\rangle$.

There are eight elements in the orbit of $\mathbf{f}+\mathbf{x}$ under the action of M_3, and we can multiply each generator of a $3B_4(i)$ by any power of $\infty_{¢}$ without changing the type, so there are $8 \times 2 \times 9 = 144$ subgroups G of A_3^* with $G \cap P' = 1$. ∎

LEMMA 5.5. *Let $A_4^* = A_4^{Bx_n}$. The group A_4^* has* 108 *subgroups G of type* $3B_4(i)$ *with* $G \cap P' = 1$.

Proof. First we note that A_4^* is stabilized by the subgroup $M_4 = \langle \chi, \rho\rho^{\chi}, \eta\rangle$ of M_{11}. By an argument similar to that of Lemma 5.4, any subcode corresponding to a subgroup of A_4^* with type $3B_4(i)$ is M_4-conjugate to $\langle \mathbf{f}+\mathbf{g}, \mathbf{x}-\mathbf{y}\rangle$ or $\langle \mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{u}, \mathbf{x}-\mathbf{y}\rangle$.

First we assume that G is a subgroup of A_4^* with type $3B_4(i)$, which contains $\infty_{\mathbf{f}+\mathbf{g}}\delta$. Now $\delta(\widetilde{\mathbf{f}+\mathbf{g}}) = 0$, so $\delta = \delta_{123}^{A}$ for $A = 0$, 1, or 2. Now $\infty_{\mathbf{f}+\mathbf{g}}\delta$ is the conjugate of $\infty_{\mathbf{f}+\mathbf{g}}$ by some power of $0_{\mathbf{f}}$. However, no conjugate of $\infty_{\mathbf{x}-\mathbf{y}}^{\pm 1}\delta_{11-12}^{B}\delta_{123}^{-B^2-1}$ by a nontrivial power of $0_{\mathbf{f}}$ is in A_4^*. Thus there are two possibilities for G, $G = \langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}^{\pm 1}\rangle$. Since there are six words in the orbit of $\mathbf{f}+\mathbf{g}$, and we can multiply each generator of G by an arbitrary power of $\infty_{¢}$ without changing its type, we see that there are 108 such subgroups of A_4^*.

Next we assume that G is a subgroup of A_4^* with type $3B_4(i)$, which contains $\infty_{\mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{u}}\delta$. Now the permutation μ maps $\mathbf{f}+\mathbf{g}$ to $\mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{u}$, $\mathbf{x}-\mathbf{y}$ to $\mathbf{x}-\mathbf{y}$, δ_{11-12} to δ_{4-7}, and δ_{123} to $\delta_{6,9,10}$. Arguing as in the previous case, we find that there are no subgroups of A_4^* with type $3B_4(i)$ that contain an element of the form $\infty_{\mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{u}}\delta$. ∎

THEOREM 5.1. *The subgroups A_3 and A_4 are not conjugate in $\mathbb{M}$.*

Proof. Let $A_i^* = A_i^{Bx_n}$ for $i = 3$ and 4. We show that A_3^* and A_4^* contain different numbers of subgroups of type $3B_4(i)$. Each subgroup G of A_3^* with type $3B_4(i)$ and $\infty_{¢} \notin G$ satisfies $G < P'$, $|G \cap P'| = 3$ and $G \cap Z(P) = 1$, or $G \cap P' = 1$. Each subgroup of $A_3^* \cap P'$ with type $3B_4(i)$ that does not contain $\infty_{¢}$ has the form $\langle i_{¢}, \delta_{123}\infty_{¢}^{\epsilon}\rangle$ or $\langle i_{¢}, \delta_{234}\infty_{¢}^{\epsilon}\rangle$, where $i = 0$, 1, or 2 and $\epsilon = 0$, 1, or 2. Thus $A_3^* \cap P'$ has 18 subgroups of type $3B_4(i)$ that do not contain $\infty_{¢}$. Adding these to the subgroups for the other two cases found in Lemmas 5.2 and 5.4, A_3^* has 162 subgroups G of type $3B_4(i)$ with $\infty_{¢} \notin G$. Similarly, $A_4^* \cap P'$ has 27 subgroups of type $3B_4(i)$ that do not contain $\infty_{¢}$, and adding this to the subgroups found in Lemmas 5.3 and 5.5 shows that A_4^* has 189 subgroups G of type $3B_4(i)$ with $\infty_{¢} \notin G$.

Since A_3^* and A_4^* have the same type and are contained in Q, they have the same number of subgroups of type $3B_4(i)$ that do contain $\infty_{¢}$. Thus A_4 has more subgroups of type $3B_4(i)$ than A_3, so they are not conjugate. ∎

5.2. *Subgroups of Type $3B_4(i)$ in A_1 and B_3*

Let $A_1^* = A_1^{Bx_n}$ and $B_3^* = B_3^{Bx_n}$. Then if $\infty_c\delta \in A_1^*$ with $\delta \in P'$, we have $c \in \langle \mathbf{x}, \mathbf{y}, \mathbf{u}\rangle$, and if $\infty_c\delta \in B_3^*$ with $\delta \in P'$, we have $c \in \langle \mathbf{x}, \mathbf{f}, \mathbf{u}\rangle$. Neither $\langle \mathbf{x}, \mathbf{y}, \mathbf{u}\rangle$ nor $\langle \mathbf{x}, \mathbf{f}, \mathbf{u}\rangle$ is conjugate under M_{11} to a subcode corresponding to the groups given in Theorems 2.1 and 2.2. Thus any subgroup of A_1^* or B_3^* of type $3B_4(i)$ must contain a nonidentity element of P'. The next lemma is an easy calculation with the Golay code.

LEMMA 5.6. *If G is a subgroup of A_1^* of type $3B_4(i)$ with $|G \cap P'| = 3$ and $G \cap Z(P) = 1$, then G contains an element $\delta\zeta$ where $\zeta \in Z(P)$ and $\delta = \delta_{123}, \delta_{1,11,12}, \delta_{1,7,10}, \delta_{147}, \delta_{156}, \delta_{189}$, or $\delta_{1,4,10}$. If G is a subgroup of B_3^* of type $3B_4(i)$ with $|G \cap P'| = 3$ and $G \cap Z(P) = 1$, then G contains an element $\delta\zeta$ where $\zeta \in Z(P)$ and $\delta = \delta_{123}, \delta_{124}, \delta_{134}$, or δ_{156}.*

THEOREM 5.2. *The subgroups A_1 and B_3 are not conjugate in $\mathbb{M}$.*

Proof. The discussion above showed that any subgroup $G < B_3^*$ of type $3B_4(i)$ contains a nonidentity element of P'. By Lemma 5.1, a subgroup $G < B_3^*$ with type $3B_4(i)$, $|G \cap P'| = 3$ and $G \cap Z(P) = 1$ must contain an element of the form $\infty_{\mathbf{f}}\infty_{\mathbf{u}}^{\alpha}\delta$ with $\delta(\mathbf{f}) = 0$, for some $\alpha = 0$, 1, or 2. As any element $\infty_{\mathbf{f}}\delta \in B_3^*$ has $\delta(\mathbf{f}) \neq 0$, the group B_3^* has no such subgroups. Thus every subgroup of B_3^* of type $3B_4(i)$ that does not contain $\infty_{¢}$ must have the form $\langle i_{¢}, \delta_{jkl}\infty_{¢}^{\epsilon}\rangle$ where $i = 0$, 1, or 2 and $\epsilon = 0$, 1, or 2, and δ_{jkl} is as described in Lemma 5.6. Thus B_3^* has 36 subgroups of type $3B_4(i)$ that do not contain $\infty_{¢}$. However, Lemma 5.6 also shows that A_1^* has at least 63 subgroups of type $3B_4(i)$ that do not contain $\infty_{¢}$. Since A_1^* and B_3^* have the same number of subgroups of type $3B_4(i)$ that contain $\infty_{¢}$, they are not conjugate. ∎

THEOREM 5.3. *The group A_5 is not conjugate to B_5, and the group B_1 is not conjugate to C_3.*

Proof. The group A_5 is conjugate to C_5. By Lemma 2.10, the $3A$ elements of C_5 span a proper subgroup of order 3^5. Let B_{5A} be the subgroup of B_5 spanned by its $3A$ elements. By Lemma 2.5, B_{5A} contains $Z(R)$ and δ_{2-3}, while by Lemma 2.10 it contains T_1. Furthermore, $T_1\infty_{\mathbf{f}}0_{\mathbf{g}}$ and $T_1\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}$ are each P-conjugate to an element of $T_1Z(P)$, so B_{5A} contains $\infty_{\mathbf{f}}0_{\mathbf{g}}$ and $\infty_{\mathbf{g}}0_{\mathbf{f}}^{-1}$. Thus the $3A$ elements of B_5 span B_5, so B_5 is not conjugate to C_5. Since A_5 is conjugate to C_5, it also is not conjugate to B_5.

Similarly, C_3 is not conjugate to C_4, since the $3A$ elements of C_3 span a proper subgroup of order 3^6, while the $3A$ elements of C_4 span the group. As B_1 is conjugate to C_4, it also is not conjugate to C_3. ∎

THEOREM 5.4. *Every maximal elementary abelian 3-subgroup of the Monster group $\mathbb{M}$ that contains a conjugate of $Z(R)$ is conjugate to A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_5, B_6, C_1, C_2, or C_3; no two of these groups are conjugate.*

Proof. Theorem 5.2 shows that A_1 and B_3 are not conjugate. Theorem 5.1 shows that A_3 and A_4 are not conjugate. Theorem 5.3 shows that A_5 and B_5 are not conjugate, and B_1 and C_3 are not conjugate. Theorem 4.1 and Lemmas 4.5 and 4.6 show that these are the only possible conjugate pairs. ∎

6. GROUPS NOT CONTAINING AN N-CONJUGATE OF $Z(R)$

We now consider the situation that A is a subgroup of $P \cap Q$, and $A \cap P'$ does not contain a subgroup of type $3B_{13}$. Since the largest $3A$ pure subgroup of P' has order 9, we have three cases to consider; $A \cap P'$ has order 9, 27, or 81. There is only one orbit of M_{11} on the possibilities for $A \cap P'$ in each case. We call these cases the $5+0$, $4+1$, and $3+2$ cases, respectively.

DEFINITION 6.1. We say that a maximal elementary abelian subgroup $A < P$ is an *$r+s$ group* if $A \cap P'$ does not contain a subgroup of type $3B_{13}$, the rank of $A/A \cap P'$ is r, and the rank of $A \cap P'/Z(P)$ is s.

The following lemma is a consequence of Eq. (1).

LEMMA 6.1. *Let A be an elementary abelian subgroup with $\infty_c\delta_c \in A$ and $\infty_d\delta_d \in A$, for $c, d \in \mathcal{G}$ and δ_c, δ_d elements of P' depending on c, d. Then $\delta_c(d) = \delta_d(c)$.*

Proof. The commutator $[\infty_c\delta_c, \infty_d\delta_d]$ is equal to $\infty_{¢}^{\delta_c(d)-\delta_d(c)}$. ∎

6.1. *The* $5+0$ *Case*

LEMMA 6.2. *Let A be an elementary abelian subgroup with $Z(P) < A < P \cap Q$ and $A \cap P' = Z(P)$. Define a map $\delta: \mathcal{G} \to \mathcal{G}^*$ by $\delta(c) = \delta_c$, where δ_c satisfies $\infty_c \delta_c \in A$. Then δ is a homomorphism from $\mathcal{G}$ to $\mathcal{G}^*$.*

We let $\mathcal{D}$ be the space of possible values of δ. By Lemma 6.1, $\mathcal{D}$ is a 15-dimensional vector space. Conjugation by elements of $P \setminus P \cap Q$ gives a 5-dimensional subspace $\mathcal{D}_0$ of $\mathcal{D}$, such that each $\delta \in \mathcal{D}$ is conjugate to every element in its coset of $\mathcal{D}_0$.

Now we recall that $\mathcal{G}$ has 22 vectors of shape $(1^6, 0^6)$. These vectors occur in 11 complementary pairs $\{x, \mathbf{u} - x\}$, and M_{11} acts 4-transitively on these pairs. Let $\tilde{\mathcal{D}}$ be an 11-dimensional vector space over $\mathbb{F}_3$ with basis elements corresponding to these pairs. Let $[x]$ denote the basis element of $\tilde{\mathcal{D}}$ corresponding to the pair $\{x, \mathbf{u} - x\}$. Define a map $\Gamma: \mathcal{D} \to \tilde{\mathcal{D}}$ by $\Gamma(\delta) = \sum_{[x]} \delta_x(x)[x]$. Since δ is a homomorphism, $\delta_{\mathbf{u}-x}(\mathbf{u} - x) = \delta_x(x)$, so Γ is well defined.

Now, every element δ of $\mathcal{D}_0$ satisfies $\delta_c(c) = 0$ for all $c \in \bar{\mathcal{G}}$ of shape $(1^6, 0^6)$. Thus $\mathcal{D}_0$ is in the kernel of Γ, so the image of Γ has dimension at most 10. We may view $\tilde{\mathcal{D}}$ as an M_{11} module, and it is isomorphic to the 11-dimensional permutation module. The image of Γ is a submodule, and by construction of an example of a group whose image in $\mathcal{D}$ is nonzero, we see that the image is the unique 10-dimensional M_{11}-submodule $\bar{\mathcal{D}}$ of $\tilde{\mathcal{D}}$. The submodule $\bar{\mathcal{D}}$ is the set of all elements of $\tilde{\mathcal{D}}$ whose components sum to 0. Thus to determine the N-classes in this case we need only determine the orbits of M_{11} on $\bar{\mathcal{D}}$.

DEFINITION 6.2. Let A be a maximal elementary abelian subgroup $A < P \cap Q$ that does not contain an N-conjugate of $Z(R)$, and let δ be the element of $\hom_{\mathbb{F}_3}(\mathcal{G}, \mathcal{G}^*)$ associated with A. The *δ-shape* of A is $1^{\alpha_1} 2^{\alpha_2} 0^{\alpha_0}$, where α_n is the number of pairs $\{x, \mathbf{u} - x\}$ of words of $\mathcal{G}$ of shape $(1^6, 0^6)$ with $\delta_x(x) = n$.

LEMMA 6.3. *The N-conjugacy classes of maximal elementary abelian subgroups $A < P \cap Q$ with $A \cap P' = Z(P)$ are in bijective correspondence with the orbits of M_{11} on $\bar{\mathcal{D}}$.*

DEFINITION 6.3. We denote the conjugacy class of a maximal elementary abelian subgroup $A < P \cap Q$ with $A \cap P' = Z(P)$ and δ-shape $1^{\alpha_1} 2^{\alpha_2} 0^{\alpha_0}$ by $E_{\alpha_1, \alpha_2, \alpha_0}$. If there is more than one conjugacy class of groups with a given δ-shape, we will denote the distinct classes by $E^a_{\alpha_1, \alpha_2, \alpha_0}$, $E^b_{\alpha_1, \alpha_2, \alpha_0}$, and so forth.

6.2. *The* $4+1$ *Case*

If $A \cap P'$ has order 27, then as $A \cap P'$ cannot have type $3B_{13}$, it must have type $3A_9B_4$. An element $\epsilon \in A \cap P' \setminus Z(P)$ corresponds to a cocode word of weight 2, whose stabilizer is isomorphic to S_5. Let $H < \mathcal{G}$ be the kernel of ϵ. We claim that A is determined by the map $c \mapsto \delta_c$, for all $c \in H$. There are five code words of shape $(1^6, 0^6)$ whose support is disjoint from the support of ϵ, and they, along with $\mathbf{u}$, span H. Let V be a 5-dimensional vector space over $\mathbb{F}_3$ that is spanned by these five code words. Then, as in the previous case, the N-classes for this case correspond to the orbits of S_5 on V.

LEMMA 6.4. *The N-conjugacy classes of maximal elementary abelian subgroups $A < P \cap Q$ with $A \cap P' = Z(P)\langle \delta_{11-12} \rangle$ are in bijective correspondence with the orbits of S_5 its 5-dimensional permutation module.*

DEFINITION 6.4. We denote the conjugacy class of a maximal elementary abelian subgroup $A < P \cap Q$ with $A \cap P' = Z(P)\langle \delta_{11-12} \rangle$ and δ-shape $1^{\alpha_1}2^{\alpha_2}0^{\alpha_0}$ by $F_{\alpha_1, \alpha_2, \alpha_0}$.

6.3. *The* $3+2$ *Case*

If $A \cap P'$ has order 81, then $A \cap P'$ has type $3A_{36}B_4$, and it corresponds to a subspace E of $\bar{\mathcal{G}}^*$ in which each element is represented by a word of weight 2.

There is a unique M_{11} orbit of such subspaces. Obviously, if E is such a subspace and contains linearly independent elements ϵ, δ, then the supports of the representatives of ϵ and δ with weight 2 are disjoint, or else E contains a word of weight 3. By 3-transitivity of M_{11}, we may assume that E contains δ_{1-2} and δ_{3-j}. Now δ_{1-2} and δ_{3-j} generate a subspace of $\bar{\mathcal{G}}^*$ containing only elements of weight 2 if and only if $4 \le j \le 9$. Furthermore, it is easy to see that T_1 and χ fix 1, 2, and 3 and are transitive on $\{4, \ldots, 9\}$.

Let $H < \mathcal{G}$ be the kernel of E. There are three code words of shape $(1^6, 0^6)$ in the kernel of E, and they span H. Let V be a 3-dimensional vector space over $\mathbb{F}_3$ that is spanned by these three code words. Then, as in the previous cases, the N-classes for this case correspond to the orbits of S_3 on V.

LEMMA 6.5. *The N-conjugacy classes of maximal elementary abelian subgroups $A < P \cap Q$ with $A \cap P' = Z(P)\langle \delta_{11-12}, \delta_{2-3} \rangle$ are in bijective correspondence with the orbits of S_3 on its 3-dimensional permutation module.*

DEFINITION 6.5. We denote the conjugacy class of a maximal elementary abelian subgroup $A < P \cap Q$ with $A \cap P' = Z(P)\langle \delta_{11-12}, \delta_{2-3} \rangle$ and δ-shape $1^{\alpha_1}2^{\alpha_2}0^{\alpha_0}$ by $G_{\alpha_1, \alpha_2, \alpha_0}$.

6.4. *Types*

LEMMA 6.6. *The type of a maximal elementary abelian subgroup $A < P \cap Q$ which does not contain an N-conjugate of $Z(R)$ is determined by its shape. Elements of class $E_{\alpha_1,\alpha_2,\alpha_0}$ have type $3A_nB_{1093-n}$, where $n = 6\alpha_0 + 6\sum_{i=0}^{2}\binom{\alpha_i}{2}$. Elements of class $F_{\alpha_1,\alpha_2,\alpha_0}$ have type $3A_nB_{1093-n}$, where $n = 54 + 18\alpha_0 + 18\sum_{i=0}^{2}\binom{\alpha_i}{2}$. Elements of class $G_{\alpha_1,\alpha_2,\alpha_0}$ have type $3A_nB_{1093-n}$, where $n = 72 + 54\alpha_0 + 54\sum_{i=0}^{2}\binom{\alpha_i}{2}$.*

Proof. Suppose that $c \in \mathcal{G}$ has shape $(1^6, 0^6)$. By Lemma 2.5, if $\delta_c(c) = 0$, then the coset $Z(P)\infty_c\delta_c$ has six elements from class $3A$ and three elements from class $3B$, while if $\delta_c(c) \neq 0$, then the coset $Z(P)\infty_c\delta_c$ has nine elements from class $3B$. Thus a group in class $E_{\alpha_1,\alpha_2,\alpha_0}$ has $6\alpha_0$ $3A$ subgroups corresponding to words of shape $(1^6, 0^6)$.

If c has shape $(1^3, 2^3, 0^6)$, then c may be written as $d - e$, where d and e each have shape $(1^6, 0^6)$. Although d and e are not unique, the pair of pairs $\{d, \mathbf{u} - d\}$ and $\{e, \mathbf{u} - e\}$ is unique. Furthermore, $\delta_c(\tilde{c}) = \delta_d(d) - \delta_e(e)$. By Lemma 2.5, if $\delta_c(\tilde{c}) = 0$, then the coset $Z(P)\infty_c\delta_c$ has three elements from class $3A$ and six elements from class $3B$, while if $\delta_c(\tilde{c}) \neq 0$, then all elements of $Z(P)\infty_c\delta_c$ are in class $3B$. Since both $d - e$ and $d - (\mathbf{u} - e)$ have shape $(1^3, 2^3, 0^6)$, there are $6\sum_{i=0}^{2}\binom{\alpha_i}{2}$ $3A$ subgroups corresponding to words of shape $(1^3, 2^3, 0^6)$.

Similar arguments apply to the groups of classes $F_{\alpha_1,\alpha_2,\alpha_0}$ and $G_{\alpha_1,\alpha_2,\alpha_0}$. ∎

7. FUSION IN $\mathbb{M}$

Our strategy for determining the fusion in $\mathbb{M}$ of the classes we have determined so far is first to show that every $5 + 0$ group fuses with a $4 + 1$ group, a $3 + 2$ group, or a group that contains $Z(R)$, then to show that every $4 + 1$ group fuses with a $3 + 2$ group, or a group containing $Z(R)$. Finally, we determine which $3 + 2$ groups fuse with groups containing $Z(R)$, and we show that the $3 + 2$ groups that do not fuse with a group containing $Z(R)$ do not fuse with each other. Conjugation by x_d shows that classes $E_{\alpha_1,\alpha_2,\alpha_0}$ and $E_{\alpha_2,\alpha_1,\alpha_0}$ are conjugate. It is implicit below that these are the same classes, and similarly for classes $F_{\alpha_1,\alpha_2,\alpha_0}$ and $G_{\alpha_1,\alpha_2,\alpha_0}$.

7.1. *Fusion of* $5 + 0$ *Groups*

LEMMA 7.1. *A group in class $E_{\alpha_1,\alpha_2,\alpha_0}$ with $\alpha_0 \geq 2$ fuses with a $4 + 1$ group, a $3 + 2$ group, or a group that contains $Z(R)$.*

Proof. By multiple transitivity of M_{11} we may assume that $\delta_{\mathbf{f}}(\mathbf{f}) = 0$ and $\delta_{\mathbf{g}}(\mathbf{g}) = 0$. Conjugation by 0_c for an appropriate c now means we may assume that $\delta_{\mathbf{f}} = 0$. Hence $\delta_{\mathbf{g}}(\mathbf{f}) = \delta_{\mathbf{f}}(\mathbf{g}) = 0$, so $\delta_{\mathbf{g}} \in \langle \delta_{123}, \delta_{2-3}, \delta_{5-6} \rangle$. Conjugation by $0_{\mathbf{f}}$ shows we may assume $\delta_{\mathbf{g}}$ is an element of $\langle \delta_{2-3}, \delta_{11-12} \rangle$. The action of the permutations ρ and $\rho\rho^{\chi}$ shows that we may assume $\delta_{\mathbf{g}} \in \langle \delta_{11-12} \rangle$. Now $B\sigma_1 B^{-1}$ maps $\infty_{\mathbf{f}-\mathbf{g}} \delta_{\mathbf{f}-\mathbf{g}}$ to an element of $\delta_{2-3} \langle \delta_{11-12} \rangle$, so A is conjugate to a subgroup of $P \cap Q$ that contains $Z(P)$ and an element of $P' \setminus Z(P)$. Thus A fuses with a $4+1$ group, a $3+2$ group, or a group containing an N-conjugate of $Z(R)$.

There are only four classes of $5+0$ groups not affected by Lemma 7.1. We show that each fuses with a $4+1$ group.

LEMMA 7.2. *There are two orbits of groups with δ-shape* $1^5 2^5 0^1$ *under the action of* M_{11}, *and one orbit each of groups of δ-shapes* $1^2 2^8 0^1$, $1^1 2^{10}$, *and* $1^4 2^7$.

Proof. Four-transitivity of M_{11} on the 11-dimensional permutation module implies that groups with the last three shapes each form a single orbit.

Next we show that M_{11}, in its action on 11 points, has two orbits on partitions into one singleton and two subsets of size five. The stabilizer of four points fixes one point and is transitive on the remaining six points, so M_{11} has two orbits on subsets of five points, one of size 66 and one of size 396. A set of five pairs of words of shape $(1^6, 0^6)$ is in the smaller orbit if there is a set of representatives from each pair that is linearly dependent. This also shows that M_{11} is transitive on partitions that have a set of size five with a linearly dependent set of representatives. If a partition has a set of size five that is linearly dependent, then the other set of size five in the partition is linearly independent. Thus there are 396 such partitions. We denote the corresponding class of subgroups $E^a_{5,5,1}$.

The stabilizer of a set of size five that does not have a linearly dependent set of representatives fixes one additional point, and is transitive on the remaining five. The orbit of size five has a linearly dependent set of representatives, so all remaining partitions are in the same orbit. There are $990 = 396 \times 5/2$ such partitions, and we denote the corresponding class of subgroups $E^b_{5,5,1}$. ∎

LEMMA 7.3. *The conjugacy class $E_{2,8,1}$ fuses with the class $F_{0,2,3}$.*

Proof. The group $Z(P)\langle \infty_{\mathbf{f}} \delta_{8-5}, \infty_{\mathbf{g}} \delta_{5-8}, \infty_{\mathbf{k}} \delta_{8-5}, \infty_{\mathbf{h}} \delta_{5-8}, \infty_{\mathbf{l}} \delta_{12-11} \rangle$ is in the conjugacy class $E_{2,8,1}$. Note that $\delta_{8-5} = \delta_{2-3}\delta_{7-4}$. The element $B\sigma_1 B^{-1}$ maps this group to

$$Z(P)\big\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{-\mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{y}} \delta_{2-3}, \infty_{-\mathbf{f}+\mathbf{g}+\mathbf{x}-\mathbf{y}} \delta_{4-7}, \infty_{\mathbf{z}+\mathbf{u}}, \infty_{\mathbf{l}} \delta_{12-11} \big\rangle,$$

which is in class $F_{0,2,3}$. ∎

LEMMA 7.4. *The conjugacy classes $E_{1,10,0}$, $E_{4,7,0}$, and the conjugacy classes $E^a_{5,5,1}$, $E^b_{5,5,1}$ are conjugate to subgroups containing $Z(R)$.*

Proof. Let $H = \langle \infty_{\mathbf{h}}\delta_{1-2}, \infty_{\mathbf{k}}\delta_{3-1}\rangle$. Then by Theorem 2.2 the group $\langle \infty_{\mathbf{¢}}\rangle H$ is conjugate to $Z(R)$. We claim that $Z(P)H\langle \infty_{\mathbf{f}}\delta_{1-12}, \infty_{\mathbf{g}}\delta_{1-11}, \infty_{\mathbf{l}}\delta_{389}\rangle$ is in conjugacy class $E_{1,10,0}$, that $Z(P)H\langle \infty_{\mathbf{f}}\delta_{2-8}, \infty_{\mathbf{g}}\delta_{2-6}, \infty_{\mathbf{l}}\delta_{134}\rangle$ is in conjugacy class $E_{4,7,0}$, that $Z(P)H\langle \infty_{\mathbf{f}}\delta_{2-8}, \infty_{\mathbf{g}}\delta_{7-10}, \infty_{\mathbf{l}}\delta_{356}\rangle$ is in conjugacy class $E^a_{5,5,1}$, and $Z(P)H\langle \infty_{\mathbf{f}}\delta_{5-9}, \infty_{\mathbf{g}}\delta_{9-5}, \infty_{\mathbf{l}}\delta_{1,5,10}\rangle$ is in conjugacy class $E^b_{5,5,1}$. ∎

7.2. *Fusion of* $4+1$ *Groups*

To determine the fusion of $4+1$ groups with groups containing $Z(R)$, we first note that each $4+1$ group is conjugate to a group containing δ_{11-12}. Although there are only two conjugacy classes of $3B_4(i)$ groups in the $\mathcal{G}$ layer under the action of N, we actually need to consider the orbits of conjugacy classes that commute with δ_{11-12}, under the action of the stabilizer of δ_{11-12} in N. It turns out that it suffices to look at groups containing $\langle \infty_{\mathbf{f+g}}, \infty_{\mathbf{y-x+f-g}}\delta_{123}\rangle$ and $\langle \infty_{\mathbf{x-y}}\delta_{123}, \infty_{\mathbf{z+f+g-u}}\delta_{2-3}\rangle$.

LEMMA 7.5. *Let A be a $4+1$ group of class $F_{0,0,5}$, $F_{1,1,3}$, $F_{0,3,2}$, $F_{1,2,2}$, $F_{0,4,1}$, $F_{1,3,1}$, $F_{2,2,1}$, $F_{1,4,0}$, or $F_{2,3,0}$. Then A fuses with a group containing $Z(R)$.*

Proof. Let A be a $4+1$ group that contains δ_{11-12}. Throughout we use Lemma 6.1. There are five pairs of words of shape $(1^6, 0^6)$ in the kernel of δ_{11-12}: **f**, **g**, **h**, **k**, and **z**. Since $\mathbf{z} = -\mathbf{f} - \mathbf{g} - \mathbf{h} - \mathbf{k}$, we have the equation $\delta_{\mathbf{z}}(\mathbf{z}) = \sum_c \sum_d \delta_c(d)$, where c and d range over **f**, **g**, **h**, and **k**.

If A contains $\langle \infty_{\mathbf{f+g}}, \infty_{\mathbf{y-x+f-g}}\delta_{123}\rangle$, then $\delta_{\mathbf{g}} = -\delta_{\mathbf{f}}$, and so $\delta_{\mathbf{g}}(\mathbf{g}) = -\delta_{\mathbf{f}}(\mathbf{g}) = -\delta_{\mathbf{g}}(\mathbf{f}) = \delta_{\mathbf{f}}(\mathbf{f})$ and $\delta_{\mathbf{z}}(\mathbf{z}) = \delta_{\mathbf{h}}(\mathbf{h}) + \delta_{\mathbf{h}}(\mathbf{k}) + \delta_{\mathbf{k}}(\mathbf{h}) + \delta_{\mathbf{k}}(\mathbf{k})$. Furthermore, $\delta_{\mathbf{h}} = \delta_{\mathbf{k}} + \delta_{\mathbf{f}} + \delta_{123}$, so $\delta_{\mathbf{h}}(\mathbf{h}) = \delta_{\mathbf{k}}(\mathbf{h}) + \delta_{\mathbf{f}}(\mathbf{h}) + 1$, and thus $\delta_{\mathbf{z}}(\mathbf{z}) = \delta_{\mathbf{h}}(\mathbf{f}) + \delta_{\mathbf{k}}(\mathbf{k}) + 1 = \delta_{\mathbf{k}}(\mathbf{f}) + \delta_{\mathbf{f}}(\mathbf{f}) + \delta_{\mathbf{k}}(\mathbf{k}) + 1$. Thus the class of A is determined by $\delta_{\mathbf{k}}(\mathbf{k})$, $\delta_{\mathbf{f}}(\mathbf{f})$, and $\delta_{\mathbf{f}}(\mathbf{k})$. Table VII gives the class of A for specific values of these quantities.

If A contains $\langle \infty_{\mathbf{x-y}}\delta_{123}, \infty_{\mathbf{z+f+g\pm u}}\delta_{2-3}\rangle$, then since $\mathbf{x} - \mathbf{y} = \mathbf{k} - \mathbf{h}$ and $\mathbf{f} + \mathbf{g} + \mathbf{z} = -\mathbf{h} - \mathbf{k}$, we see that $\delta_{\mathbf{k}} = \delta_{3-1}$ and $\delta_{\mathbf{h}} = \delta_{1-2}$. Thus $\delta_{\mathbf{h}}(\mathbf{h}) = 2$, $\delta_{\mathbf{k}}(\mathbf{k}) = 1$, and $\delta_{\mathbf{h}}(\mathbf{f}) = \delta_{\mathbf{h}}(\mathbf{g}) = \delta_{\mathbf{k}}(\mathbf{f}) = \delta_{\mathbf{k}}(\mathbf{g}) = 0$. Now $\delta_{\mathbf{z}}(\mathbf{z}) = \delta_{\mathbf{f}}(\mathbf{f}) + \delta_{\mathbf{g}}(\mathbf{g}) - \delta_{\mathbf{f}}(\mathbf{g})$, so the class of A is determined by $\delta_{\mathbf{f}}(\mathbf{f})$, $\delta_{\mathbf{g}}(\mathbf{g})$, and $\delta_{\mathbf{z}}(\mathbf{z})$. Again see Table VII for the class of specific groups. ∎

Now we show that each $4+1$ group that we have not shown to fuse with a group containing $Z(R)$ fuses with a $3+2$ group.

LEMMA 7.6. *The following classes of groups fuse*:

1. *Class $F_{0,2,3}$ fuses with class $G_{0,2,1}$.*
2. *Class $F_{0,1,4}$ fuses with class $G_{0,3,0}$.*
3. *Class $F_{0,5,0}$ fuses with class $G_{0,1,2}$.*

TABLE VII
$4+1$ Groups that Contain a Conjugate of $Z(R)$

$A \supset \langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}+\mathbf{f}-\mathbf{g}}\delta_{123}\rangle$				$A \supset \langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$			
$\delta_{\mathbf{f}}(\mathbf{f})$	$\delta_{\mathbf{k}}(\mathbf{k})$	$\delta_{\mathbf{f}}(\mathbf{k})$	Class of A	$\delta_{\mathbf{f}}(\mathbf{f})$	$\delta_{\mathbf{g}}(\mathbf{g})$	$\delta_{\mathbf{z}}(\mathbf{z})$	Class of A
0	0	2	$F_{0,0,5}$	0	0	0	$F_{1,1,3}$
0	2	0	$F_{1,1,3}$	0	0	2	$F_{1,2,2}$
0	2	2	$F_{0,3,2}$	0	1	2	$F_{2,2,1}$
1	2	2	$F_{2,3,0}$	0	2	2	$F_{1,3,1}$
2	0	2	$F_{0,2,3}$	1	2	2	$F_{2,3,0}$
2	2	0	$F_{1,3,1}$	2	2	2	$F_{1,4,0}$

Proof. Let $G(s,t) = Z(P)\langle \delta_{2-3}, \delta_{11-12}, \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{f}-\mathbf{g}}\delta_{7-4}^{t}, \infty_{\mathbf{z}}\delta_{123}^{s}\rangle$. The conjugate of $G(s,t)$ by $B\sigma_1 B^{-1}$ is the group

$$F(s,t) = Z(P)\langle \delta_{11-12}, \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{f}-\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}^{t}\delta_{2-3}, \infty_{\mathbf{z}}\delta_{123}^{s}\rangle.$$

It is an easy computation to show that for $t \neq 0$, $G(s,t)$ has shape $t^2 s$ and $F(s,t)$ has shape $(s-t)^2 s 0^2$. For $s=0$ and $t=2$, this implies that $G_{0,2,1}$ fuses with $F_{0,2,3}$, and for $s=2$ and $t=2$, this implies that $G_{0,3,0}$ fuses with $F_{0,1,4}$.

Furthermore, let

$$F = Z(P)\big\langle \delta_{11-12}, \infty_{\mathbf{f}}\delta_{7-4}, \infty_{\mathbf{g}}\delta_{4-7}, \infty_{\mathbf{h}}\delta_{123}^{-1}, \infty_{\mathbf{k}}^{-1}\delta_{123}\big\rangle.$$

Then F is in class $F_{0,5,0}$, and the image of F under $B\sigma_1 B^{-1}$ is

$$Z(P)\big\langle \delta_{11-12}, \delta_{4-7}, \infty_{\mathbf{h}}\delta_{3-1}, \infty_{\mathbf{k}}\delta_{2-1}, \infty_{\mathbf{z}}\delta_{123}^{-1}\big\rangle,$$

which is in class $G_{0,1,2}$. ∎

7.3. *Fusion of* $3+2$ *Groups*

To determine fusion of $3+2$ groups with groups that contain $Z(R)$, note that each $3+2$ group is conjugate to one containing δ_{11-12} and δ_{4-7}. There are only two relevant classes of $3B_4(i)$ groups, with representatives as in the $5+0$ case.

LEMMA 7.7. *Let A be a $3+2$ group that fuses with a group containing $Z(R)$. Then A is conjugate in N to a group that contains δ_{11-12} and δ_{4-7}, and one of the $3B_4(i)$ groups $\langle \infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}\delta_{123}\rangle$ or $\langle \infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$.*

Proof. By 3-transitivity, we may assume that A contains δ_{11-12} and δ_{4-j} for some j. Now the pointwise stabilizer of $\{4, 11, 12\}$ has two orbits on the remaining points, $\{1, 8, 9\}$ and $\{2, 3, 5, 6, 7, 10\}$. Since $\delta_{11-12} - \delta_{4-1} \equiv \delta_{7,10,12}$, A must be conjugate in N to a group containing δ_{11-12} and δ_{4-7}.

Next we show that A is conjugate to a group with a subgroup of type $3B_4(i)$ that contains $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$. The stabilizer in M_{11} of $\langle\delta_{11-12}, \delta_{4-7}\rangle$, which contains T_2 and ρ^{σ_1}, has three orbits on the cosets of $\langle\mathbf{u}\rangle$ in the kernel of $\langle\delta_{11-12}, \delta_{4-7}\rangle$. The orbits and code words they contain are $O_1 \supset \{\mathbf{h}, \mathbf{k}, \mathbf{z}\}$, $O_2 \supset \{\mathbf{h} \pm \mathbf{k}, \mathbf{h} \pm \mathbf{z}, \mathbf{k} \pm \mathbf{z}\}$, and $O_3 \supset \{\mathbf{h} \pm \mathbf{k} \pm \mathbf{z}\}$.

By Lemma 2.2, if A has a subgroup of type $3B_4(i)$ that contains an element that corresponds to a code word of shape $(1^6, 0^6)$, then it contains elements that correspond to two such words. The orbit O_1 above shows we may assume that the two words are $\mathbf{h}$ and $\mathbf{k}$, so the subgroup of type $3B_4(i)$ contains an element that corresponds to $\mathbf{k} - \mathbf{h} = \mathbf{x} - \mathbf{y}$. Now conjugation by an appropriate element of the form 0_c implies that the subgroup of type $3B_4(i)$ contains $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$.

If A has a subgroup of type $3B_4(i)$ that does not contain an element that corresponds to a code word of shape $(1^6, 0^6)$, then it must contain elements corresponding to code words from both orbits O_2 and O_3. Examining the shapes of the code words in such a subgroup shows that it is conjugate to a group that has a subgroup of type $3B_4(i)$ that contains $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$.

Of the 19 subgroups of type $3B_4(i)$ in the $\mathscr{G}$-layer that contain $\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}$, only those containing elements that correspond to $\mathbf{f} + \mathbf{g}$ or $\mathbf{z} \pm (\mathbf{f} + \mathbf{g} - \mathbf{u})$ are in the kernel of δ_{11-12} and δ_{4-7}. Since the permutation $\rho^{\sigma_1}\rho^{\chi\sigma_1}$ of M_{11} switches the subgroups generated by $\mathbf{f} + \mathbf{g}$ and $\mathbf{z} - \mathbf{f} - \mathbf{g} + \mathbf{u}$, we need only consider the $3 + 2$ groups containing $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{f}+\mathbf{g}}\rangle$ or $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}-\mathbf{u}}\delta_{2-3}\rangle$. ∎

LEMMA 7.8. *If A is a $3+2$ group that fuses with a group containing $Z(R)$, then A is in class $G_{1,1,1}$ or $G_{1,2,0}$.*

Proof. From the previous lemma, we may assume that A contains one of the groups $\langle\infty_{\mathbf{f}+\mathbf{g}}, \infty_{\mathbf{y}-\mathbf{x}}\delta_{123}\rangle$ or $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{z}+\mathbf{f}+\mathbf{g}-\mathbf{u}}\delta_{2-3}\rangle$. We assume that A is a group of the form $Z(P)\langle\delta_{11-12}, \delta_{4-7}, \infty_{\mathbf{h}}\delta_{\mathbf{h}}, \infty_{\mathbf{k}}\delta_{\mathbf{k}}, \infty_{\mathbf{z}}\delta_{\mathbf{z}}\rangle$ for some $\delta_{\mathbf{h}}$, $\delta_{\mathbf{k}}$, and $\delta_{\mathbf{z}} \in P'$. Suppose that A contains the group $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{f}+\mathbf{g}}\rangle$. Since $\mathbf{f} + \mathbf{g} = -\mathbf{h} - \mathbf{k} - \mathbf{z}$ and $\mathbf{x} - \mathbf{y} = \mathbf{k} - \mathbf{h}$, we see that $\delta_{\mathbf{k}} = \delta_{\mathbf{h}} + \delta_{123}$. Thus $\delta_{\mathbf{k}}(\mathbf{k}) = \delta_{\mathbf{h}}(\mathbf{k}) + 1 = \delta_{\mathbf{k}}(\mathbf{h}) + 1 = \delta_{\mathbf{h}}(\mathbf{h}) + 2$. Also, since $\delta_{\mathbf{h}} + \delta_{\mathbf{k}} + \delta_{\mathbf{z}} = 0$, we see that $\delta_{\mathbf{z}}(\mathbf{z}) = -\delta_{\mathbf{h}}(\mathbf{z}) - \delta_{\mathbf{k}}(\mathbf{z}) = -\delta_{\mathbf{z}}(\mathbf{h}) - \delta_{\mathbf{z}}(\mathbf{k}) = \delta_{\mathbf{h}}(\mathbf{h}) + \delta_{\mathbf{k}}(\mathbf{h}) + \delta_{\mathbf{h}}(\mathbf{k}) + \delta_{\mathbf{k}}(\mathbf{k}) = \delta_{\mathbf{h}}(\mathbf{h}) + 1$. Thus the only $3 + 2$ groups that are conjugate to a group that contains $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{f}+\mathbf{g}}\rangle$ are in class $G_{1,1,1}$.

Suppose that A contains $\langle\infty_{\mathbf{x}-\mathbf{y}}\delta_{123}, \infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}-\mathbf{u}}\delta_{2-3}\rangle$. Since $\mathbf{f} + \mathbf{g} + \mathbf{z} = -\mathbf{h} - \mathbf{k}$ and $\mathbf{x} - \mathbf{y} = \mathbf{k} - \mathbf{h}$, we see that $\delta_{\mathbf{k}} = \delta_{3-1}$ and $\delta_{\mathbf{h}} = \delta_{1-2}$. Thus $\delta_{\mathbf{k}}(\mathbf{k}) = 1$ and $\delta_{\mathbf{h}}(\mathbf{h}) = 2$, so A is in class $G_{1,1,1}$ or class $G_{1,2,0}$. ∎

7.4. *Nonfusion of Remaining* $3+2$ *Classes*

We have shown that every maximal elementary abelian group that does not contain a conjugate of $Z(R)$ is conjugate to a group in one of the classes $G_{0,0,3}$, $G_{0,2,1}$, $G_{0,1,2}$ or $G_{0,3,0}$. These classes have distinct types except for the last two, which have type $3A_{234}B_{859}$. We show that these two classes are not conjugate by showing that groups from these classes have different numbers of subgroups of type $3A_4$.

LEMMA 7.9. *A subgroup in conjugacy class* $G_{0,3,0}$ *has* 1053 *subgroups of type* $3A_4$.

Proof. The group $G_1 = Z(P)\langle \delta_{2-3}, \delta_{11-12}, \infty_{\mathbf{f}}\delta_{268}, \infty_{\mathbf{g}}\delta_{268}, \infty_{\mathbf{z}}\delta_{268}\rangle$ is in conjugacy class $G_{0,3,0}$. Now $G_1 \cap P'$ contains the nine subgroups $\langle \delta_{2-3}0_{¢}^{i}, \delta_{11-12}0_{¢}^{j}\rangle$ for $i, j = 0, 1$, or 2.

Let A be a subgroup of type $3A_4$ of G_1 that contains a nonidentity element of P', but is not contained in P'. By Lemma 2.5, A contains an element of one of the 10 cosets $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{z}}$, $(A \cap P')\infty_{\mathbf{g}\pm\mathbf{z}}$, $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}}$, or $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}\pm\mathbf{z}}$. Lemma 2.6 eliminates the possibility that A contains an element of $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}\pm\mathbf{z}}$. If A contains an element of $(A \cap P')\infty_{\mathbf{f}-\mathbf{g}}$, after conjugation by an appropriate element of $P'\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$, we may assume that $\infty_{\mathbf{f}-\mathbf{z}} \in A$. Now by Lemma 2.6, if $A = \langle \infty_{\mathbf{f}-\mathbf{z}}, \delta\rangle$, then $\delta = \pm\delta_{5-6}$ or $\pm\delta_{11-12}$. Moreover, no subgroup of the form $\langle \infty_{\mathbf{f}-\mathbf{z}}, \infty_{\mathbf{u}}\delta\rangle$ has type $3A_4$. Thus there are two subgroups of G_1 of type $3A_4$ with nontrivial intersection with P' that contain $\infty_{\mathbf{f}-\mathbf{z}}$. A similar argument applies to subgroups that contain $\infty_{\mathbf{f}+\mathbf{z}}$, $\infty_{\mathbf{f}\pm\mathbf{g}}$, and $\infty_{\mathbf{g}\pm\mathbf{z}}$. Including conjugates by elements of $\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ gives a total of 36 such subgroups.

Now let A be a subgroup of type $3A_4$ of G_1 with $A \cap P' = 1$. Using Lemma 2.5 we see that A must contain an element from one of the cosets $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}\pm\mathbf{z}}$. If A contains an element of $(A \cap P')\infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}}$, after conjugation by an appropriate element of $P'\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ we may assume that $\infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}+\mathbf{u}} \in A$. Then A is one of the groups $\langle \infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}+\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}}\delta^{\pm}_{2-3}\rangle$, $\langle \infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}+\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}+\mathbf{u}}\delta^{\pm}_{8-9}\rangle$, or $\langle \infty_{\mathbf{f}+\mathbf{g}+\mathbf{z}+\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}-\mathbf{u}}\delta^{\pm}_{5-6}\rangle$. A similar argument applies to subgroups that contain $\infty_{\mathbf{f}+\mathbf{g}-\mathbf{z}}$, $\infty_{\mathbf{f}-\mathbf{g}+\mathbf{z}}$, and $\infty_{\mathbf{f}-\mathbf{g}-\mathbf{z}}$. Including conjugates by elements of $\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ gives a total of 72 such subgroups.

Finally, we observe that if $\langle g, h\rangle$ is a subgroup of type $3A_4$, then so is $\langle g\infty_{¢}^{i}, h\infty_{¢}^{j}\rangle$ for $i, j = 0, 1$, or 2. Thus each of the subgroups listed above determines a total of nine subgroups of type $3A_4$ with the same image in $G_1/\langle \infty_{¢}\rangle$. This gives a total of $9 \times (9 + 36 + 72) = 1053$ subgroups of G_1 of type $3A_4$. ∎

LEMMA 7.10. *A subgroup in conjugacy class* $G_{0,1,2}$ *has* 1161 *subgroups of type* $3A_4$.

Proof. This proof is an adaptation of the proof of Lemma 7.9. The group $G_2 = Z(P)\langle \delta_{2-3}, \delta_{11-12}, \infty_{\mathbf{f}}, \infty_{\mathbf{g}}, \infty_{\mathbf{z}}\delta_{123}^{-1}\rangle$ is in class $G_{0,1,2}$, and it contains the nine subgroups $\langle \delta_{2-3}0_{¢}^{i}, \delta_{11-12}0_{¢}^{j}\rangle$ for $i, j = 0, 1$, or 2.

Let A be a subgroup of type $3A_4$ of G_2 that contains a nonidentity element of P' but is not contained in P'. Then A contains an element of one of the cosets $(A \cap P')\infty_{\mathbf{f}}$, $(A \cap P')\infty_{\mathbf{g}}$, or $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}}$. If A contains an element of $(A \cap P')\infty_{\mathbf{f}}$, after conjugation by an appropriate element of $P'\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ we may assume that $\infty_{\mathbf{f}} \in A$ or $\infty_{\mathbf{f}-\mathbf{u}} \in A$. Using Lemma 2.6, we see that if $A = \langle \infty_{\mathbf{f}}, \delta\rangle$ or $A = \langle \infty_{\mathbf{f}-\mathbf{u}}, \infty_{\mathbf{u}}\delta\rangle$, then $\delta = \pm\delta_{8-9}$ or $\pm\delta_{11-12}$; if $A = \langle \infty_{\mathbf{f}-\mathbf{u}}, \delta\rangle$, or $A = \langle \infty_{\mathbf{f}}, \infty_{\mathbf{u}}\delta\rangle$, then $\delta = \pm\delta_{2-3}$ or $\pm\delta_{5-6}$. A similar argument applies to subgroups that contain an element of $(A \cap P')\infty_{\mathbf{g}}$, and including conjugates by elements of $\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ gives a total of 36 such subgroups.

If A contains an element of $(A \cap P')\infty_{\mathbf{f}-\mathbf{g}}$, after conjugation by an appropriate element of $P'\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ we may assume that $\infty_{\mathbf{f}-\mathbf{g}} \in A$. Again using Lemma 2.6, we see that $A = \langle \infty_{\mathbf{f}-\mathbf{g}}, \delta\rangle$, where $\delta = \pm\delta_{2-3}$ or $\pm\delta_{11-12}$. A similar argument applies to subgroups that contain an element of $(A \cap P')\infty_{\mathbf{f}+\mathbf{g}}$, and including conjugates by elements of $\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ gives a total of 12 such subgroups.

Now let A be a subgroup of type $3A_4$ of G_2 with $A \cap P' = 1$. Then A must contain an element from each of the cosets $(A \cap P')\infty_{\mathbf{f}\pm\mathbf{g}}$. Conjugation by an appropriate element of $P'\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ shows that we may assume that $\infty_{\mathbf{f}+\mathbf{g}-\mathbf{u}} \in A$. Then A is conjugate by P' to one of the groups $\langle \infty_{\mathbf{f}+\mathbf{g}-\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}}\delta^{\pm}_{2-3}\rangle$, $\langle \infty_{\mathbf{f}+\mathbf{g}-\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}}\delta^{\pm}_{11-12}\rangle$, $\langle \infty_{\mathbf{f}+\mathbf{g}-\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}+\mathbf{u}}\delta^{\pm}_{8-9}\rangle$, or $\langle \infty_{\mathbf{f}+\mathbf{g}-\mathbf{u}}, \infty_{\mathbf{f}-\mathbf{g}-\mathbf{u}}\delta^{\pm}_{5-6}\rangle$. Including conjugates by elements of $\langle 0_{\mathbf{x}}, 0_{\mathbf{y}}\rangle$ gives a total of 72 such subgroups.

As with G_1, we note that each of the subgroups listed above determines a total of nine subgroups of type $3A_4$ with the same image in $G_2/\langle \infty_{¢}\rangle$. This gives a total of $9 \times (9 + 36 + 12 + 72) = 1161$ subgroups of G_2 of type $3A_4$. ∎

LEMMA 7.11. *Classes $G_{0,3,0}$ and $G_{0,1,2}$ do not fuse in $\mathbb{M}$.*

Proof. Lemmas 7.9 and 7.10 show that elements of classes $G_{0,3,0}$ and $G_{0,1,2}$ have different numbers of subgroups of type $3A_4$, so they cannot fuse in $\mathbb{M}$. ∎

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 5.4 determines the classes of subgroups that contain $Z(R)$. The discussion at the beginning of Section 6 shows that any other group is conjugate to a $5+0$ group, a $4+1$ group, or a $3+2$ group. Lemmas 7.1, 7.3, 7.4, 7.5, and 7.6 show that each $5+0$ group and each $4+1$ group is conjugate to a $3+2$ group or a group that contains $Z(R)$. Lemmas 7.8 and 7.11, along with the types of the $3+2$ groups, show that there are four distinct classes of $3+2$ groups that do not contain a conjugate of $Z(R)$ (Table VIII). ∎

TABLE VIII
Classes of subgroups without a conjugate of $Z(R)$

Class	$G_{0,0,3}$	$G_{0,2,1}$	$G_{0,1,2}$	$G_{0,3,0}$
Type	$3A_{396}B_{697}$	$3A_{180}B_{913}$	$3A_{234}B_{859}$	$3A_{234}B_{859}$

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